

An effective Kronecker-Weyl equidistribution theorem and an application to generalized Skewes' numbers

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Summary

- 1 Oscillations and Chebyshev's bias
- 2 The Kronecker-Weyl theorem
- 3 Effective Linear Independence and generalized Skewes' numbers

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- In other words,

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In particular, $\pi(x) - \text{li}(x)$ changes sign infinitely many times.

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 - How often $\psi(x) - x$ or $\pi(x) - \text{li}(x)$ is positive or negative?
 - How often does it change sign?
 - When does it change sign for the first time?

Number of sign changes

- Let $V_1(T)$ and $V_3(T)$ denote the number of times $\pi(x) - \text{li}(x)$ and $\psi(x) - x$ respectively changes sign in the interval $[1, T]$.

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Theorem (Pólya, Ingham, Levinson, Kaczorowski...).

For $i \in \{1, 3\}$, there exists an (astronomically) small effective constant $\delta_i > 0$ such that

$$\liminf_{T \rightarrow \infty} \frac{V_i(T)}{\log T} \geq \frac{\gamma_0}{\pi} + \delta_i,$$

where $\gamma_0 \approx 14.13 \dots$ is the imaginary part of the smallest non-trivial zero of ζ .

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- This is very far from the expected $C\sqrt{T}$!

Skewes' number: first sign change

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- On the other hand, we only know $x_S \geq 10^{19}$ (Büthe, 2015). This is quite the gap!

Frequency of positivity: density

Theorem (Rubinstein-Sarnak, 1994).

Assume RH and the imaginary parts of the non-trivial zeros of ζ are linearly independent over \mathbb{Q} (LI). Then the **logarithmic density**

$$\delta := \lim_{X \rightarrow \infty} \frac{1}{\log X} \int_2^X \mathbf{1}_{\pi(t) < \text{li}(t)} \frac{dt}{t} = \lim_{X \rightarrow \infty} \frac{1}{X} \int_{\log 2}^X \mathbf{1}_{\pi(e^t) < \text{li}(e^t)} dt$$

exists. Moreover,

$$\delta \approx 0.99999973 \dots \neq 1.$$

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- Note that under RH the **natural density**

$$\lim_{X \rightarrow \infty} \frac{1}{X} \int_2^X \mathbf{1}_{\pi(t) < \text{li}(t)} dt$$

does **not** exist! (Kaczorowski, 1993)

Sketch of proof I.

- Let $E(x) := \frac{\pi(e^x) - \text{li}(e^x)}{e^{x/2}/x}$. Under RH, the explicit formula yields

$$E(x) = -\log(2\pi) - \sum_{\gamma} \frac{e^{i\gamma x}}{\frac{1}{2} + i\gamma} + O\left(\frac{1}{x}\right),$$

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and the probability measures on \mathbb{R} characterised by

$$\mu_X(f) = \frac{1}{X} \int_0^X f(E(t)) dt$$

and

$$\mu_X^{(T)}(f) = \frac{1}{X} \int_0^X f(E^{(T)}(t)) dt$$

for a bounded continuous $f : \mathbb{R} \rightarrow \mathbb{C}$.

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for any **bounded continuous** $f : \mathbb{R} \rightarrow \mathbb{C}$. Using the LI hypothesis, μ is a continuous distribution and we can approximate $\mathbf{1}_{\pi(et) < \text{li}(et)}$ by $f(E(t))$ with bounded Lipschitz functions f .

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$$\delta_{q;a,b} := \lim_{X \rightarrow \infty} \frac{1}{\log X} \int_2^X \mathbf{1}_{\pi(t;q,a) > \pi(t;q,b)} \frac{dt}{t}$$

under GRH and LI for the Dirichlet L -functions modulo q .

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- Moreover, they showed that

$$\begin{cases} \delta_{q;a,b} = \frac{1}{2} & \text{if } a \text{ and } b \text{ are both squares or non-squares mod } q, \\ \delta_{q;a,b} < \frac{1}{2} & \text{if } a \text{ is a square mod } q \text{ and } b \text{ is not.} \end{cases}$$

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- This is the explanation of **Chebyshev's bias**: squares tend to lose versus non-squares in a "prime number race". This is explained by the constant term in the explicit formula which shifts the mean of the limiting distribution $\mu_{q;a,b}$ towards its sign.

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Theorem (B.-Hayani-Untrau, 2026).

Let $A > 1$. Assume GRH and ELI_A for Dirichlet L -functions modulo q . Then

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- Actually we get bounds on the first sign change in more general "prime number races", for example the "squares vs non-squares mod q " race.

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Classical Kronecker-Weyl

Theorem (Kronecker-Weyl equidistribution).

Let $\gamma_1, \dots, \gamma_n$ be real numbers. Then

$$\Gamma = \{(e^{i\gamma_1 x}, \dots, e^{i\gamma_n x}) \mid x \in \mathbb{R}^+\} \subseteq \mathbb{T}^n$$

is **equidistributed** in a subtorus $\bar{\Gamma}$ of \mathbb{T}^n of dimension $\dim \text{Span}_{\mathbb{Q}}(\gamma_1, \dots, \gamma_n)$.

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- In other words, for any continuous function $g : \mathbb{T}^n \rightarrow \mathbb{C}$,

$$\nu_X(g) := \frac{1}{X} \int_0^X g(e^{i\gamma_1 t}, \dots, e^{i\gamma_n t}) dt \xrightarrow{X \rightarrow \infty} \int_{\bar{\Gamma}} g d\nu,$$

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- This may also be formulated in the following way: the probability measures ν_X on \mathbb{T}^n converge weakly to ν as $X \rightarrow \infty$.

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- In particular, if $\gamma_1, \dots, \gamma_n$ are linearly independent over \mathbb{Q} , then $\bar{\Gamma} = \mathbb{T}^n$ and ν is the Lebesgue measure, so $(e^{i\gamma_1 x}, \dots, e^{i\gamma_n x})$ behaves asymptotically on average like a vector of independent uniform random variables on the unit circle \mathbb{T} .
- It is classically proved first for characters $(z_1, \dots, z_n) \mapsto z_1^{m_1} \dots z_n^{m_n}$ of \mathbb{T}^n and extended by density. In particular, **there is no explicit rate of convergence** to the limit.

Wasserstein metric

To measure the rate of convergence, we will use:

Definition (Dual formulation of Wasserstein metric).

Let π_1, π_2 be Borel probability measures on a polish space M with finite first moment. Their 1-**Wasserstein distance** is defined by

$$W_1(\pi_1, \pi_2) = \sup_{f \in 1\text{-Lip}(M, \mathbb{R})} |\pi_1(f) - \pi_2(f)|.$$

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Theorem (Bobkov-Ledoux, 2020, Kowalski-Untrau, 2025).

Let π_1, π_2 be Borel probability measures on \mathbb{T}^n with finite first moment. Then for any $H > 0$,

$$W_1(\pi_1, \pi_2) \ll \frac{\sqrt{n}}{H} + \left(\sum_{\substack{m \in \mathbb{Z}^n \\ 1 \leq \|m\|_\infty \leq H}} \frac{|\widehat{\pi}_1(m) - \widehat{\pi}_2(m)|^2}{\|m\|_2^2} \right)^{1/2}.$$

Metric Kronecker-Weyl

Corollary.

Let $\gamma_1, \dots, \gamma_n$ be real numbers. For any $X > 0$, let ν_X be the probability measure on \mathbb{T}^n characterised by $\nu_X(g) = \frac{1}{X} \int_0^X g(e^{i\gamma_1 t}, \dots, e^{i\gamma_n t}) dt$ for any continuous $g : \mathbb{T}^n \rightarrow \mathbb{C}$ and let ν be the Haar measure on $\bar{\Gamma}$. Then for any $H > 0$,

$$W_1(\nu_X, \nu) \ll \frac{\sqrt{n}}{H} + \frac{1}{X} \left(\sum_{\substack{m \in \mathbb{Z}^n \\ \|m\|_\infty \leq H, \langle m, \gamma \rangle \neq 0}} \frac{1}{\|m\|_2^2 |\langle m, \gamma \rangle|^2} \right)^{1/2},$$

where $\langle m, \gamma \rangle = \sum_{j=1}^n m_j \gamma_j$.

Summary

- 1 Oscillations and Chebyshev's bias
- 2 The Kronecker-Weyl theorem
- 3 **Effective Linear Independence and generalized Skewes' numbers**

Lower bounds

- Going back to the proof of Rubinstein-Sarnak, we see that if we want to make the Kronecker-Weyl step effective, we need upper bounds on $\langle m, \gamma \rangle^{-1}$, *i.e.* lower bounds on $\langle m, \gamma \rangle$, with γ being a vector of imaginary parts of zeros of Dirichlet L -functions.

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Conjecture (ELI_A for $A > 1$).

Let $q \geq 2$ and let \mathcal{X}_q be a set of Dirichlet characters modulo q . We let $(\gamma_n)_{n \geq 1}$ be an enumeration in non-decreasing order of the positive imaginary parts of zeros, counted with multiplicities, of Dirichlet L -functions associated with the characters $\chi \in \mathcal{X}_q$ and for any $T > 0$, $N(T) = \sum_{\gamma_n \leq T} 1$. Then for $T > 0$,

$$|\langle m, \gamma \rangle_T| := \left| \sum_{j=1}^{N(T)} m_j \gamma_j \right| \gg_A N(T)^{-N(T)^A}$$

for any integers $m_1, \dots, m_{N(T)}$, not all zero, satisfying $|m_j| \leq N(T)$.

About ELI_A

- A similar (stronger) hypothesis was recently used by Lamzouri to obtain

$$\psi(x) - x = \Omega_{\pm}(x^{1/2}(\log \log \log x)^2),$$

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- It is heuristically justified by assuming a good distribution of the γ_i 's mod 1 and using the pigeonhole principle.
- We can draw a parallel with measures of transcendency: if ξ is a real number, then a measure of how much it is transcendent can be read on lower bounds of the form

$$|P(\xi)| = \left| \sum_{i=0}^n m_i \xi^i \right| \gg H^{-f(n)},$$

where $P \in \mathbb{Z}[T]$ has coefficient satisfying $|m_1|, \dots, |m_n| \leq H$ and $f(n)$ is some positive quantity.

Back to measures

- This time, let

$$E(t) = \varphi(q) \frac{\pi(e^t; q, a) - \pi(e^t; q, b)}{e^{t/2}/t}$$

and $E^{(T)}(t) =$ its explicit formula truncated at height T . We consider the measures

$$\mu_X(f) = \frac{1}{X} \int_0^X f(E(t)) dt, \mu_X^{(T)}(f) = \frac{1}{X} \int_0^X f(E^{(T)}(t)) dt$$

and $\mu^{(T)}$ the limit of $\mu_X^{(T)}$ as $X \rightarrow \infty$ (Kronecker-Weyl).

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- From estimates on zeros, for $X \geq \log T$, $W_1(\mu_X, \mu_X^{(T)}) \ll \varphi(q)^2 \log q \left(\frac{\log T}{\sqrt{T}} + \frac{1}{\sqrt{X}} \right)$.

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- Letting $X \rightarrow \infty$ and by completeness, $W_1(\mu, \mu^{(T)}) \ll \varphi(q)^2 \log q \frac{\log T}{\sqrt{T}}$.
- Using ELI_A , we get for $X \geq N_q(T)^{2N_q(T)^A}$, $W_1(\mu_X^{(T)}, \mu^{(T)}) \ll_A \frac{\varphi(q)^2 \log q}{N_q(T)^{1/2}}$.

Bounds on W_1 and the logarithmic integral

Optimising the value of T in terms of q and X , we get:

Theorem (B.-Hayani-Untrau, 2026).

Let $A > 1$. Assume GRH and ELI_A for Dirichlet L -functions modulo q . For X (quantifiably) large enough, we have

$$W_1(\mu, \mu_X) \ll_A (\varphi(q)^2 \log q)^{\frac{3}{2} + \frac{1}{2A}} (\log \log X)^2 (\log X)^{-\frac{1}{2A}}.$$

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Letting $\mathcal{P}_{q;a,b} := \{x \geq 2 \mid \pi(x; q, a) > \pi(x; q, b)\}$, we deduce:

Theorem (B.-Hayani-Untrau, 2026).

Under the same hypotheses,

$$\left| \frac{1}{X} \int_0^X \mathbf{1}_{\mathcal{P}_{q;a,b}}(e^t) dt - \delta_{q;a,b} \right| \ll_A (\varphi(q)^2 \log q)^{\frac{3}{4} + \frac{1}{4A}} (\log \log X) (\log X)^{-\frac{1}{4A}}.$$

End of proof

- We remark that $\delta_{q;a,b} > c$ where $c > 0$ is an absolute constant because $\delta_{q;a,b} \xrightarrow{q \rightarrow \infty} \frac{1}{2}$ uniformly in (a, b) .

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- As soon as $\int_{\log 2}^X \mathbf{1}_{\mathcal{P}_{q;a,b}}(e^t) dt > 0$, we know there is a $y \leq e^X$ such that $\pi(y; q, a) > \pi(y; q, b)$, which yields the final bound

$$\log \log \log x_{q;a,b} \ll_A \log \varphi(q).$$

Squares vs. non-squares

- The same strategy applies to more general prime number races. For example, let

$$R_q := \{x^2 \mid x \in (\mathbb{Z}/q\mathbb{Z})^\times\} \text{ and } NR_q := (\mathbb{Z}/q\mathbb{Z})^\times \setminus R_q$$

and

$$\rho(q) := \#\{x \in (\mathbb{Z}/q\mathbb{Z})^\times \mid x^2 = 1\},$$

so that $\#R_q = \frac{\varphi(q)}{\rho(q)}$ and $\#NR_q := \#R_q(\rho(q) - 1)$.

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- Let $\pi(x; q, R) := \sum_{\substack{p \leq x \\ p \in R_q}} 1$ and $\pi(x; q, NR) := \pi(x) - \pi(x; q, R)$. We want to compare $(\rho(q) - 1)\pi(x; q, R)$ and $\pi(x; q, NR)$ (expecting the latter to be ahead in the race): let

$$\delta_{q;R,NR} := \lim_{X \rightarrow \infty} \frac{1}{X} \int_{\log 2}^X \mathbf{1}_{(\rho(q)-1)\pi(e^t; q, R) > \pi(e^t; q, NR)} dt$$

and

$$x_{q;R,NR} := \inf\{x \geq 2 \mid (\rho(q) - 1)\pi(x; q, R) > \pi(x; q, NR)\}.$$

Squares vs. non-squares

- Following the method of Rubinstein-Sarnak, Fiorilli (2014) proved that this race can be **extremely biased**: if $\frac{\rho(q)}{\log \text{rad}(q)} \rightarrow \infty$ then $\delta_{q;R,NR} \rightarrow 0$.

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Theorem (B.-Hayani-Untrau, 2026).

Let $(q_n)_n$ be a sequence of integers such that $\frac{\rho(q_n)}{\log \text{rad}(q_n)} \xrightarrow{n \rightarrow \infty} \infty$ and assume GRH and ELI_A for some $A > 1$ for Dirichlet L -functions of quadratic characters modulo each q_n . Then

$$\log \log \log x_{q_n;R,NR} \ll_A \frac{\rho(q_n)}{\log \text{rad}(q_n)} + \log \rho(q_n).$$

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$$\log \log \log x_{q_n;\text{R,NR}} \ll_A \frac{\rho(q_n)}{\log \text{rad}(q_n)} + \log \rho(q_n).$$

- Moreover, Fiorilli's conjecture is **false** and the term $\log \rho(q_n)$ is important: for each $L > 0$, we construct unconditionnally a sequence $(q_n)_n$ such that $\log \log \log x_{q;\text{R,NR}} > L \frac{\rho(q)}{\log \text{rad}(q)} \rightarrow \infty$.

Over function fields

- Another context where this method can be applied is the count of irreducible polynomials over finite fields.

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no need for $ELI_A!$

L -functions over function fields

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$$L(s, \chi) = \sum_{A \in \mathbb{F}_q[T] \text{ monic}} \frac{\chi(A)}{|A|^s}$$

is a polynomial $\mathcal{L}(u, \chi) \in \mathbb{Z}[T]$ in the variable $u := q^{-s}$, which admits a factorization

$$\mathcal{L}(u, \chi) = \prod_{j=1}^d (1 - \alpha_j u) \times P(u)$$

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- Moreover, the inverse roots α_j are of the form

$$\alpha_j = \sqrt{q} e^{i\gamma_j}$$

with $\gamma_j \in [0, 2\pi]$. (Riemann Hypothesis over function fields)

Baker bound

- Noting that the γ_j 's are **logarithms of algebraic numbers**, Baker's theory of logarithmic forms applies and provides a substitute for ELI_A !

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Theorem (B.-Hayani-Untrau, 2026+).

Let q be a prime power, $Q \in \mathbb{F}_q[T]$ be monic of degree $h \geq 1$ and $g = \lfloor \frac{h-1}{2} \rfloor$. Let \mathcal{D} be a set of r Dirichlet characters modulo Q and $\gamma = (\gamma_1, \dots, \gamma_n)$ be an enumeration of the distinct arguments in $[0, \pi]$ of the non-trivial inverse roots of the Dirichlet L -functions attached to the elements of \mathcal{D} . Then for any $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, we have

$$|\langle m, \gamma \rangle| \geq (rh \|m\|_\infty)^{-c(q,r,h)},$$

where

$$c(q, r, h) = (32rh)^{rh+3} (2^g g!)^{r^2 g + 3r} (\log q)^{rg}.$$

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- We can then obtain explicit bounds for Wasserstein metrics for irreducible polynomial races. The difficulty now lies in getting (almost unconditional) lower bounds for the density...

Thank you for your attention!