The Lambda function and prime numbers

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Abstract

We study the Lambda function (obviously !) of Von Mangoldt and other greek letters, and we show how they can be used to prove the Prime number Theorem®.

1 Introduction : the π function

The most important mathematical function is certainly (maybe not) the prime counting function π, *i.e.* for all $x \geq 2$,

$$\pi(x) := \#\{p \leq x\},$$

where here and below p will denote that the index set is the set of prime numbers. The asymptotic behaviour of this function has focused, and is still focusing, the attention of many mathematicians. The prime number theorem states that

$$\pi(x) \underset{x\to+\infty}{\sim} \frac{x}{\log x},$$

or in other words

$$\lim_{x\to+\infty} \frac{\pi(x)\log x}{x} = 1.$$

It is a non-trivial result with a rich history (which was told in a previous Lambda seminar [1]), and whose proof we are going to sketch.

The symbol log will denote the usual logarithm, and the principal branch of the complex logarithm on $\mathbb{C}\setminus\mathbb{R}^-$, and we recall that $f(x) = O(g(x))$ means there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ holds for every relevant values of x.

2 From π to ψ, through θ

2.1 The θ function

One can write $\pi(x) = \sum_{p\leq x} 1$. For a reason that should be clear later, it is more natural to count prime numbers p with a weight $\log p$. Let us introduce the θ function of Chebyshev (1821-1894) : for all $x \geq 2$,

$$\theta(x) := \sum_{p\leq x} \log p.$$

Note that, for all $x \geq 2$, $\theta(x)$ is simply the logarithm of the *primorial*

$$\prod_{p \leq x} p.$$

In 1848, Chebyshev showed by elementary methods the following remarkable result : there exist constants $c_1, c_2 > 0$ such that for all $x \geq 2$,

$$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}.$$

On this occasion, he introduced the θ function, and the ψ function which we will talk about soon.

Clearly, we havee $\theta(x) \leq \pi(x) \log x$. We are going to see that we can compare those quantities more precisely.

Lemma 2.1. *For all $x \geq 2$, we have*

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} \,\mathrm{d}t.$$

Proof. This is an example of *summation by parts*, the discrete analog of integration by parts. Let us show the following general result : if $(a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers and $f : [0, +\infty[\longrightarrow \mathbb{C}$ is class $\mathcal{C}^1$ function then for every $x \geq 0$,

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_0^x A(t)f'(t) \,\mathrm{d}t,$$

where $A(x) := \sum_{n \leq x} a_n$. To show this, we use Abel summation, *i.e.* we write $a_n = A(n) - A(n-1)$ (with the convention $A(x) = 0$ for $x < 0$) :

$$\begin{aligned}
\sum_{n \leq x} a_n f(n) = \sum_{n \leq x} (A(n) - A(n-1))f(n) &= \sum_{n \leq x} A(n)f(n) - \sum_{n \leq x-1} A(n)f(n+1) \\
&= A(\lfloor x \rfloor)f(\lfloor x \rfloor) + \sum_{n \leq x-1} A(n)(f(n) - f(n+1)) \\
&= A(\lfloor x \rfloor)f(\lfloor x \rfloor) - \sum_{n \leq x-1} A(n) \int_n^{n+1} f'(t) \,\mathrm{d}t \\
&= A(\lfloor x \rfloor)f(\lfloor x \rfloor) - \int_0^{\lfloor x \rfloor} A(t)f'(t) \,\mathrm{d}t
\end{aligned}$$

since A is constant on every interval of the form $[n, n+1[$. There only remains to observe that $A(x)f(x) = A(\lfloor x \rfloor)f(\lfloor x \rfloor) + \int_{\lfloor x \rfloor}^x A(t)f'(t) \,\mathrm{d}t$ for the same reason.

Let us come back to our lemma. It is enough to apply the summation by parts formula to the sequence defined by

$$a_n = \begin{cases} \log p \text{ if } n \text{ is a prime number } p \\ 0 \text{ otherwise} \end{cases},$$

whose summatory function is θ, and $f = \frac{1}{\log}$ (note that $\theta(t) = 0$ for $t < 2$). $\square$

We easily deduce[1] from this lemma that if $\theta(x) \underset{x\to+\infty}{\sim} x$ then $\pi(x) \underset{x\to+\infty}{\sim} \frac{x}{\log x}$, that is, the prime number theorem. This is the estimate we will be looking for.

2.2 The ψ function

Let us now introduce the following function : for all $x \geq 2$, let

$$\psi(x) = \sum_{k\geq 1} \sum_{p^k \leq x} \log p.$$

This is some kind of generalization of the θ function, but we also take into consideration the powers of prime numbers. One can see that, for every $x \geq 2$, $\psi(x)$ is the logarithm of the LCM of the positive integers $\leq x$. For all $k \geq 1$ and prime number p, the conditions $p^k \leq x$ and $p \leq x^{1/k}$ are equivalent, so that

$$\psi(x) = \sum_{k=1}^{+\infty} \theta(x^{1/k}).$$

Note that for every $x \geq 2$, this sum is in fact finite : $\theta(x^{1/k})$ is zero as soon as $x^{1/k} < 2$, that is when $k > \lfloor \frac{\log x}{\log 2} \rfloor$.

We will now show that if $\psi(x) \underset{x\to+\infty}{\sim} x$ then $\theta(x) \underset{x\to+\infty}{\sim} x$. To do this, we will need the following lemma.

Lemma 2.2. *We have*

$$\theta(x) = O(x)$$

for $x \geq 2$.

Proof. We use a clever trick : for all integer $n \geq 1$, the bnomial coefficient $\binom{2n}{n}$ is divisible by every p such that $n < p \leq 2n$. Indeed, one has $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ and if $n < p \leq 2n$, then p divides $(2n)!$ but not $n!$. By Gauss' lemma, we obtain

$$\left(\prod_{n<p\leq 2n} p\right) \mid \binom{2n}{n}.$$

Since $\binom{2n}{n} \leq 4^n$ (look at the expansion of $(1+1)^{2n}$), we thus have

$$\prod_{n<p\leq 2n} p \leq 4^n.$$

Taking logarithms, we find

$$\theta(2n) - \theta(n) \leq n \log 4.$$

[1] $\int_2^x \frac{\mathrm{d}t}{(\log t)^2} \mathrm{d}t = \int_2^{x^{1/2}} \frac{\mathrm{d}t}{(\log t)^2} \mathrm{d}t + \int_{x^{1/2}}^x \frac{\mathrm{d}t}{(\log t)^2} \mathrm{d}t = O(x^{1/2}) + O\left(\frac{x}{(\log x)^2}\right) = o\left(\frac{x}{\log x}\right)$

Observe that, in fact,

$$\theta(2x) - \theta(x) \leq x \log 4$$

for every real $x \geq 2$ since the left-hand side does not change when we replace x by its integer part, and the right-hand side is an increasing function of x. To conclude, there simply remains to sum those telescopic inequalities :

$$\begin{aligned}\theta(x) &= \theta(x) - \theta(x/2) + \theta(x/2) - \theta(x/4) + \theta(x/4) - \ldots \\ &\leq \left(x + \frac{x}{2} + \frac{x}{2^2} + \ldots\right) \log 4 = 2x \log 4.\end{aligned}$$

□

We can now write

$$0 \leq \psi(x) - \theta(x) = \sum_{k=2}^{\lfloor \frac{\log x}{\log 2} \rfloor} \theta(x^{1/k}) = O(x^{1/2} \log x),$$

by upper bounding each $\lfloor \frac{\log x}{\log 2} \rfloor - 1 = O(\log x)$ terms by $O(x^{1/2})$, which implies that if $\psi(x) \underset{x\to+\infty}{\sim} x$ then $\theta(x) \underset{x\to+\infty}{\sim} x$, and that is what we are now going to show.

3 The Λ function

Let us now write the ψ function with a summation over all integers. To do this, we introduce the Lambda function (finally !) of Von Mangoldt (1854-1925) : for every $n \in \mathbb{N}$,

$$\Lambda(n) := \begin{cases} \log p \text{ if } n \text{ is a power of a prime number } p \\ 0 \text{ otherwise} \end{cases}.$$

Therefore, we have for every $x \geq 2$,

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

To study the behaviour of a sequence, it is usual to study a generating function attached to it. One could for example study the function

$$z \mapsto \sum_{n=0}^{+\infty} \Lambda(n) z^n$$

ou her cousin

$$z \mapsto \sum_{n=0}^{+\infty} \frac{\Lambda(n)}{n!} z^n,$$

but actually, it is preferable to work with Dirichlet (1805-1859) series in this context. Thus, we introduce the function

$$F : s \mapsto \sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s}.$$

It is clear that for $n \geq 1$, $\Lambda(n) \leq \log n$, so that the above series converges asbolutely when $\mathfrak{Re}(s) > 1$. Since the convergence is norma on every half-plane of the form $\Omega_a := \{s \in \mathbb{C} \mid \mathfrak{Re}(s) > a\}$, with $a > 1$, the function F is holomorphic on Ω_1.

In the case of a power series $f(z) = \sum_{n\geq 0} a_n z^n$, we know we can extract the coefficient a_n with an integral formula, Cauchy's (1789-1857) formula :

$$a_n = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f(z)}{z^{n+1}} \, \mathrm{d}z,$$

where $\mathcal{C}$ is a sufficiently smooth closed curve looping around the origin. Can we do the same for Dirichlet series ? The answer is given by Perron's (1880-1975) formula, which we give here in a simplified form.

Lemma 3.1 (Perron's formula)**.** *Let*

$$f(s) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$$

be a Dirichlet series which converges absolutely for $\mathfrak{Re}(s) \geq c > 0$. Then for every $x \geq 1$ not an integer, we have

$$\sum_{n\leq x} a_n = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} f(s) \frac{x^s}{s} \, \mathrm{d}s.$$

Remark. If $x = N$ is an integer, one has to take $\frac{a_N}{2}$ as the last term of the sum.

Proof. We only give an idea of the proof. We start by showing that

$$\frac{1}{2i\pi} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, \mathrm{d}s = \begin{cases} 1 \text{ si } y > 1 \\ 0 \text{ si } 0 < y < 1 \end{cases} + O\left(\frac{y^c}{T|\log y|}\right),$$

by using the residue theorem (when $y > 1$ we integrate on a rectangle around 0 and send its left side to infinity, when $0 < y < 1$ we integrate on a rectangle not containing 0 and send its right side to infinity, the error term is the contribution of the horizontal sides). We now sum those equalities, taking $y = \frac{x}{n}$, which yields

$$\sum_{n\leq x} a_n = \sum_{n=1}^{+\infty} \frac{1}{2i\pi} \int_{c-iT}^{c+iT} a_n \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s} + o_{T\to+\infty}(1).$$

Finally, the normal convergence in the domain $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) \geq c\}$ allows us to permute the sum and the integral. $\square$

We have now established the very useful formula

$$\psi(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} \, \mathrm{d}s,$$

for every $x \geq 1$ not an integer and every $c > 1$. For integer x, we saw in the previous remark that there is a missing factor of $\frac{\Lambda(x)}{2}$, which is clearly $O(\log x)$. Therefore, we have

$$\psi(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} \, \mathrm{d}s + O(\log x)$$

for every $x \geq 2$ and $c > 1$. The asymptotic estimate we are after is x, so we can forget about this extra term. We now need to identify the function F.

4 The relation between F and ζ

The infamous Riemann (1826-1866) zeta function, defined for $\mathfrak{Re}(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

now comes into play. Its usefulness lies in its factorization as *Euler* (1707-1783) *product* :

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

which is a formal evidence when we expand each factor as a geometric series. The convergence of the series $\sum_p \log\left(1 - \frac{1}{p^s}\right)$ on Ω_1 implies that ζ does not vanish on this domain, and one can take its complex logarithm. We have, still for $\mathfrak{Re}(s) > 1$,

$$\log \zeta(s) = \sum_p - \log\left(1 - \frac{1}{p^s}\right) = \sum_p \sum_{k=1}^{+\infty} \frac{1}{kp^{ks}}.$$

Keeping an eye on convergence issues, we get by differentiating

$$(\log \zeta)'(s) = -\sum_p \sum_{k=1}^{+\infty} \frac{\log p}{p^{ks}}.$$

We now observe that the coefficient in front of $\frac{1}{n^s}$ in the above Dirichlet series is simply $-\Lambda(n)$. In other words, we established that the analytic function F is actually

$$-(\log \zeta)' = -\frac{\zeta'}{\zeta}.$$

We have at our disposal Von Mangoldt's formula :

$$\psi(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, \mathrm{d}s + O(\log x),$$

for every $x \geq 2$ and $c > 1$. Let us recall that our goal is to estimate the size of the quantity $\psi(x)$, more precisely to show that $\psi(x) \underset{x \to +\infty}{\sim} x$. To do this, we going to use the residue theorem once again, to get an estimate on the size of this integral. One has to ask about the location of poles of the integrand. We previously observed that the ζ function has no zero on Ω_1, so there is no pole in sight for the function $s \mapsto -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$, defined on Ω_1. However, as Riemann showed in 1859, this function admits an analytic continuation to the whole complex plane. Riemann actually shows that ζ has a meromorphic continuation to $\mathbb{C}$, with only one simple pole at 1, with residue 1. We will be content with the following computation : by summation by parts, we have

$$\sum_{1\leq n\leq x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s\int_1^x \frac{\lfloor t \rfloor}{t^{s+1}}\,\mathrm{d}t.$$

Thus, when $\mathfrak{Re}(s) > 1$, we get, by taking the limit in x,

$$\zeta(s) = s\int_1^{+\infty} \frac{\lfloor t \rfloor}{t^{s+1}}\,\mathrm{d}t = s\int_1^{+\infty} \frac{\mathrm{d}t}{t^s} - s\int_1^{+\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t = \frac{s}{s-1} - s\int_1^{+\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t$$

and this last expression clearly defines a meromorphic function on Ω_0, with a simple pole at 1, with residue 1.

Let us now look at the poles of the function $s \mapsto -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$ inside Ω_0. The pole of ζ at 1 provides a pole with residue x and each zero ρ (counted with multiplicity) of ζ contributes to a pole, with residue $-\frac{x^\rho}{\rho}$. More estimates from complex analysis and the residue theorem allow us to show that

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s = x - \sum_\rho \frac{x^\rho}{\rho} + O_c(1),$$

where the summation is taken over the set of zeros ρ of ζ satisfying $0 < \mathfrak{Re}(\rho) \leq 1$. It now appears to be crucial to locate the zeros of ζ to conclude the proof.

Theorem 4.1 (Hadamard, De la Vallée-Poussin, 1896)**.** *The ζ function does not vanish on the line* $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) = 1\}$.

This argument is at the core of the original proof by Hadamard (1865-1963) and De la Vallée-Poussin (1866-1962) of the prime number theorem (and later in the proofs of Wiener (1894-1964) and Ikehara (1904-1984)). In our current argumentation, it would be in fact necessary to establish a *zero-free region* for the ζ-function. There could be zeros of the ζ function with real parts accumulating to 1, making the sum $\sum_\rho \frac{x^\rho}{\rho}$ too large and not negligible compared to the main term x (recall that the absolute value of x^ρ is $x^{\mathfrak{Re}(\rho)}$). The zero-free region of De la Vallée-Poussin allows us to prove

$$\psi(x) = x + O\left(x\exp(-c\sqrt{\log x})\right),$$

where $c > 0$ is a constant, which completes the proof of

$$\psi(x) \underset{x\to+\infty}{\sim} x,$$

and as we have seen of

$$\pi(x) \underset{x\to+\infty}{\sim} \frac{x}{\log x}.$$

5 Conclusion

We have shown the links between the different counting functions π, θ and ψ, and how to obtain the prime number theorem using results on the location of zeros of the Riemann ζ function. At the core of this was the Von Mangoldt Λ function.

There exist differents proofs of the prime number theorem, but each of them involves the Λ function in one way or another. It is also involved in the proofs of other types of prime number theorem, such as the prime number theorem in arithmetic progressions, the theorem of Chebotarev (1894-1947).

The smallest error term possible in the estimate $\psi(x) \underset{x \to +\infty}{\sim} x$ is $O(x^{1/2} \log x)$. It corresponds to the *Riemann hypothesis* : the zeros ρ of ζ such that $0 < \mathfrak{Re}(\rho) < 1$ have real part $1/2$. We are far from being able to proving this : the best zero-free region for the ζ function hasn't been improved since 1958 (Korobov (1917-2004) and Vinogradov (1891-1983)) and it doesn't even exclude the possibility of zeros with real parts accumulating to 1. For more details and developments on this, we recommand the excellent [2].

References

[1] Alexandre Bailleul, *Autour du théorème des nombres premiers* (2018). `https://www.math.u-bordeaux.fr/~abailleul/Lambda.pdf`.

[2] Hugh L. Montgomery and Robert C. Vaughan, *Multiplicative number theory I: Classical theory*, Vol. 97, Cambridge University Press, 2007.

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