The prime number theorem - Part 2

In this exercise sheet we finish proving (by admitting some technical steps) the prime number theorem, in the following form :

$$\pi(x) = \mathrm{Li}(x) + O\left(x \exp\left(-c(\log x)^a\right)\right),$$

where

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$$

and $a, c > 0$ are constants.

Exercise 1. [$\mathrm{Li}(x)$ is a better approximation than $\frac{x}{\log x}$]

1. By integrating by parts, show that $\mathrm{Li}(x) \underset{x\to+\infty}{\sim} \frac{x}{\log x}$.
2. Show that the prime number theorem in the form above implies

$$\pi(x) = \sum_{k=1}^{N} \frac{(k-1)!x}{(\log x)^k} + O\left(\frac{x}{(\log x)^{N+1}}\right)$$

for any $N \geq 1$.

Exercise 2. [The Perron formula]

For $c, T, y > 0$, let

$$\delta(y) = \begin{cases} 1 \text{ if } y > 1 \\ \frac{1}{2} \text{ if } y = 1 \\ 0 \text{ if } y < 1 \end{cases}$$

and

$$I_c(T, y) = \frac{1}{2i\pi} \int_{c-iT}^{c+iT} \frac{y^s}{s} \, \mathrm{d}s.$$

1. By using the residue theorem, prove that for $y \neq 1$,

$$|I_c(T, y) - \delta(y)| \leq \frac{y^c}{\pi T |\log y|}$$

and that

$$|I_c(T, 1) - \delta(1)| < \frac{c}{\pi T}.$$

2. Show that if $(a_n)_{n\geq 1}$ is a complex sequence, $F(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ its Dirichlet series with abscissa of absolute convergence σ_a, then for every $c > \max(\sigma_a, 0)$ and $x > 0$, one has Perron's formula

$$\sum_{n\leq x}{}' a_n = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} \, \mathrm{d}s,$$

where $\sum'_{n\leq x} a_n = \sum_{n<x} a_n + \frac{1}{2} a_x$ and $a_x = 0$ if $x \notin \mathbb{N}$. In particular,

$$\psi(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, \mathrm{d}s + O(\log x).$$

Remark. Define the Mellin transform of a suitable function f by

$$\mathcal{M}_f(s) = \int_0^{+\infty} f(x)x^{s-1}\,\mathrm{d}x.$$

By a simple change of variable, it is easy to see that, as a function of $t = \mathfrak{Im}(s)$, this is the Fourier transform of a function related to f, and one can show that Perron's formula is a consequence of the Fourier inversion formula.

Exercise 3. [Zero-free regions and end of the proof]

We now admit the following facts which would require a bit more time to prove :

i) The Perron formula with $c = 1+\frac{1}{\log x}$ and truncating the integral to imaginary parts between $-T$ and T, combined with the residue theorem lead to

$$\psi(x) = x - \sum_{\substack{\rho=\beta+i\gamma,\zeta(\rho)=0\\ 0\leq\beta\leq 1,|\gamma|\leq T}} \frac{x^\rho}{\rho} + O\left(\frac{x\log^2 x}{T}\right)$$

for $2 \leq T \leq x$.

ii) General results on the distribution of zeros of holomorphic functions (Jensen's formula) imply that for $T \geq 2$, the number of zeros $\rho = \beta + i\gamma$ of ζ with $0 \leq \beta \leq 1, T \leq \gamma \leq T+1$ is $O(\log T)$.

1. Let $\delta \in]0, 1/2[$. Assume that ζ has no zero $\rho = \beta + i\gamma$ with $\beta \geq 1-\delta$. Show that $\psi(x) = x + O(x^{1-\delta}(\log x)^2)$ (*Hint : Split the sum in i) in intervals of length* 1 *for* γ, use *ii*) and choose T wisely). Deduce that $\pi(x) = \mathrm{Li}(x) + O(x^{1-\delta}\log x)$.

   Remark. This is unknown as of today for **any** such δ. The Riemann Hypothesis is about knowing this for **all** such δ. The case $\delta = 0$ actually suffices to prove the prime number theorem, but without an error term.

2. Combining the trigonometric trick of TD 11, Exercise 5 with some more estimates on ζ, one can prove the following : There exists $c > 0$ such that ζ has no zero $\rho = \beta + i\gamma$ with $|\gamma| \geq 2$ and $\beta \geq 1 - \delta(\gamma)$, where $\delta(\gamma) = \frac{c}{\log|\gamma|}$.

   Show that $\psi(x) = x + O\left(x\exp\left(-c'\sqrt{\log x}\right)\right)$ and deduce that $\pi(x) = \mathrm{Li}(x) + O\left(x\exp\left(-c''\sqrt{\log x}\right)\right)$ for some $c', c'' > 0$.

Remark. In 1958, Korobov and Vinogradov proved a larger zero-free region, with $\delta(\gamma) = \frac{c}{(\log|\gamma|)^{2/3}(\log\log|\gamma|)^{1/3}}$, which implies that

$$\pi(x) = \mathrm{Li}(x) + O\left(x\exp\left(-c'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right).$$

This is still the best result as of today.

Exercise 4. [Consequences of the prime number theorem]

1. Let p_n denote the n^{th} prime number. Show that $p_n \underset{n\to+\infty}{\sim} n\log n$.

2. Let $\varepsilon > 0$. Show that if x is large enough, then there exists a prime number in $[x, (1+\varepsilon)x]$ (the case $\varepsilon = 1$ is called Bertrand's postulate and was proved by Chebyshev).

3. Prove that
$$\sum_{p\le x}\frac{1}{p} = \log\log x + C + o\left(\frac{1}{\log x}\right),$$
where C is a constant.
4. Deduce that the average number of prime factors of n is about $\log\log n$, *i.e.*
$$\frac{1}{x}\sum_{n\le x}\omega(n) \underset{x\to+\infty}{\sim} \log\log x,$$
where $\omega(n)$ is the number of prime factors of n.
5. Prove that $\limsup_{n\to+\infty}\frac{\omega(n)\log\log n}{\log n} = 1$. Thus $\omega(n)$ is close to $\frac{\log n}{\log\log n}$ infinitely often. (*Hint : What kind of numbers could maximize their number of prime factors with respect to their size ?)*
6. Let $\pi_2(x)$ denote the number of $n \le x$ which are the product of two distinct prime numbers. Show that
$$\pi_2(x) = \sum_{p\le\sqrt{x}}\pi\left(\frac{x}{p}\right) + O\left(\frac{x}{(\log x)^2}\right)$$
and deduce that
$$\pi_2(x) = \frac{x\log\log x}{\log x} + O\left(\frac{x}{\log x}\right).$$

Exercise 5. [The prime ideal theorem in $\mathbb{Q}(\sqrt{d})$]

The prime ideal theorem, proved by Landau in 1903, states that if K is a number field and $\pi_K(x) = |\{\mathfrak{p} \text{ prime ideal of } \mathcal{O}_K \mid N(\mathfrak{p}) \le x\}|$, then
$$\pi_K(x) \underset{x\to+\infty}{\sim} \frac{x}{\log x}.$$
In fact, $\pi_K(x) \underset{x\to+\infty}{\sim} \pi_{K,1}(x)$, where $\pi_{K,1}(x) = |\{\mathfrak{p} \text{ prime ideal of } \mathcal{O}_K \mid f(\mathfrak{p}) = 1, N(\mathfrak{p}) = p \le x\}|$.

1. (a) Show that for $x \ge 2$, $\pi_{\mathbb{Q}(\sqrt{-1})}(x) = 2\pi(x;4,1) + \pi(x^{1/2};4,3) + 1$, where $\pi(x;q,a) = |\{p \le x \mid p \equiv a \bmod q\}|$.
   (b) Let us admit the prime number theorem in arithmetic progressions : for a prime to q, $\pi(x;q,a) \underset{x\to+\infty}{\sim} \frac{1}{\varphi(q)}\frac{x}{\log x}$. Prove the prime ideal theorem for $\mathbb{Q}(\sqrt{-1})$.
2. Let $d \ne 0, 1$ be squarefree and $K = \mathbb{Q}(\sqrt{d})$. Prove the prime ideal theorem in K by generalizing the above question.

Remark. The quadratic reciprocity law was used in the general case above. More generally, if $K/\mathbb{Q}$ is an abelian extension, then splitting of primes in K are also determined by congruences, but this lies much deeper as this uses Artin's reciprocity law, which is a vast generalization of quadratic reciprocity.

Exercise 6. [The prime number theorem over finite fields]

Let $P_q(n) = \{P \in \mathbb{F}_q[X] \mid P \text{ irreducible}, \deg P = n\}$ and $\pi_q(n) = |P_q(n)|$.

1. Show that
$$X^{q^n} - X = \prod_{d|n} \prod_{P \in P_q(d)} P.$$
2. Prove that $q^n = \sum_{d|n} \pi_q(d)$.
3. Deduce that $\pi_q(n) \underset{n \to +\infty}{\sim} \frac{q^n}{n}$. (*Hint : Recall the Möbius inversion formula. If* $f(n) = \sum_{d|n} g(d)$ *then* $g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$.)
4. The size of the polynomial $P \in \mathbb{F}_q[X]$ is $|P| = q^{\deg P}$ (it is the cardinality of the quotient $\mathbb{F}_q[X]/(P)$). Prove the prime number theorem in $\mathbb{F}_q[X]$:
$$|\{P \in \mathbb{F}_q[X] \mid P \text{ irreducible}, |P| \leq x\}| \underset{x \to +\infty}{\sim} \frac{x}{\log_q(x)}.$$

Remark. In fact we obtain the Riemann Hypothesis for $\mathbb{P}^1(\mathbb{F}_q)$ in this case :
$$|\{P \in \mathbb{F}_q[X] \mid P \text{ irreducible}, |P| \leq x\} = \frac{x}{\log_q(x)} + O(x^{1/2+\varepsilon})$$
for any $\varepsilon > 0$. In a more general context, the Riemann Hypothesis over finite fields has been proved by Weil in the 40's for curves, and by Deligne in the 70's for varieties, but this is extremely deep.