The prime number theorem - Part 1

The goal of this exercise sheet and the next is to prove the prime number theorem : If $\pi(x) = \#\{p \leq x \mid p \text{ prime}\}$, then

$$\pi(x) \underset{x\to+\infty}{\sim} \frac{x}{\log x}.$$

This takes many steps and relies on properties of the Riemann zeta function, defined by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

for $\mathfrak{Re}(s) > 1$.

In the following, the letter p always denotes a prime number, a summation over $n \leq x$ means a summation over $\{n \in \mathbb{N} \mid n \leq x\}$ or $\{n \in \mathbb{N}^* \mid n \leq x\}$ depending on context, and log denotes the natural logarithm. We also recall that log admits a principal determination on $\mathbb{C} \setminus \mathbb{R}^-$ which is a right inverse of the exponential function and which satisfies

$$\log(1+z) = \sum_{n\geq 1} \frac{(-1)^{n+1} z^n}{n}$$

for $|z| < 1$.

Exercise 1. [Chebyshev's functions]

For $x \geq 2$, we let $\theta(x) = \sum_{p\leq x} \log p$ and $\psi(x) = \sum_{p,k\geq 1, p^k \leq x} \log p$.

1. Show that for every integer $n \geq 1$, $\theta(n) = \log P^{\#}(n)$, where $P^{\#}(n) = \prod_{p\leq n} p$, and $\psi(n) = \log \operatorname{lcm}(1, \ldots, n)$.
2. Let $n \geq 1$. Show that the binomial coefficient $\binom{2n}{n}$ is divisible by every prime p such that $n < p \leq 2n$.
3. Deduce that for any $n \geq 1$, $\theta(2n) - \theta(n) \leq n \log 4$, and use it to show that $\theta(x) \leq x \log 4$ for $x \geq 2$.
4. Prove that $\psi(x) - \theta(x) = O(x^{1/2})$.
5. Let $(a_n)_n$ be a complex sequence and $f \in \mathcal{C}^1([0, +\infty[)$. For $t \in \mathbb{R}$, write $A(t) = \sum_{n\leq t} a_n$ (with the convention $A(t) = 0$ for $t < 0$). Prove that
$$\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_0^x A(t)f'(t)\,\mathrm{d}t.$$
(*Hint : Write* $a_n = A(n) - A(n-1)$)
6. Deduce that
$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t}\,\mathrm{d}t.$$

7. By splitting the integral in two, show that

$$\int_2^x \frac{\mathrm{d}t}{\log^2 t} = O\left(\frac{x}{\log^2 x}\right).$$

8. Prove that the prime number theorem is equivalent to

$$\psi(x) \underset{x\to+\infty}{\sim} x.$$

Remark. Chebyshev proved in 1852 that $\psi(x) \asymp x$, *i.e.* $\psi(x) = O(x)$ and $x = O(\psi(x))$. As a consequence, $\pi(x) \asymp \frac{x}{\log x}$. He even proved that if $\frac{\pi(x)\log x}{x}$ admits a limit at infinity, it must be 1, but proving that this limit exists is the hard part...

Exercise 2. [The Von Mangoldt function]
We define the Von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^k \\ 0 \text{ otherwise} \end{cases}$$

for every $n \in \mathbb{N}$.
Let Ω be the half-plane $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > 1\}$.

1. Show that ψ is the summatory function Λ ,*i.e.* $\psi(x) = \sum_{n\le x} \Lambda(n)$.
2. Let F be the Dirichlet series of Λ, *i.e.*

$$F(s) = \sum_{n\ge 1} \frac{\Lambda(n)}{n^s}.$$

Compute the abscissa of convergence of F.

3. Recall the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $s \in \Omega$. Prove that $\zeta(s) \neq 0$ for $s \in \Omega$. (*Hint : It suffices to prove it is the exponential of a complex number*).

4. By using the Euler product, expand $\log\zeta(s)$ into a Dirichlet series, and identify the function F. (*Hint : To use the functional equation of the logarithm, check that two analytic functions on Ω coincide on* $]1, +\infty[$)
5. Assuming for now that ζ admits an analytic continuation to $\mathbb{C}$, with only a simple pole at 1, classify the poles of F and give their orders and residues.

Exercise 3. [The functional equation of zeta]
Recall the Gamma function is defined by

$$\Gamma(s) = \int_0^{+\infty} e^{-t} x^{s-1}\,\mathrm{d}x$$

for $\mathfrak{Re}(s) > 0$. By integrating by parts, one shows that $\Gamma(s+1) = s\Gamma(s)$.

1. Show that Γ admits a meromorphic continuation to $\mathbb{C}$, with simple poles at each $-k$ and residue $\frac{(-1)^k}{k!}$, for $k \in \mathbb{N}$.

2. Let $s \in \mathbb{C}$ such that $\mathfrak{Re}(s) > 0$ and $n \in \mathbb{N}^*$. Show that

$$\frac{\Gamma(s/2)}{n^s} = \pi^{s/2} \int_0^{+\infty} e^{-\pi n^2 y} y^{s/2} \frac{\mathrm{d}y}{y}.$$

3. Deduce that for $\mathfrak{Re}(s) > 1$, one has

$$I(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^{+\infty} \left(\frac{\theta(t)-1}{2}\right) t^{s/2} \frac{\mathrm{d}t}{t}$$

where $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$.

4. We admit the functional equation $\theta(1/t) = \sqrt{t}\theta(t)$ for $t > 0$ (this is an application of the Poisson summation formula). Show that $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + f(s) + f(1-s)$, where

$$f(s) = \int_1^{+\infty} \left(\frac{\theta(t)-1}{2}\right) t^{s/2} \frac{\mathrm{d}t}{t}.$$

5. Deduce that I extends to a meromorphic function on $\mathbb{C}$ with simple pole at 0 and 1 and satisfying $I(s) = I(1-s)$.
6. Prove that ζ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at 1 and "find" its zeros.

Exercise 4. [Elementary estimates on ζ]

In this exercise, the complex variable is denoted by $s = \sigma + it$. We will provide upper bounds on ζ in different regions of the half-plane $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > 0\}$.

1. Let $\delta > 0$. Show that for $\sigma \geq 1+\delta$, one has $|\zeta(s)| \leq \zeta(1+\delta)$. In particular, ζ is bounded in any half-plane of the form $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) \geq 1+\delta\}$.
2. Use partial summation to prove that for $1 \leq x < y$ and $s \in \mathbb{C}$,

$$\sum_{x<n\leq y} \frac{1}{n^s} = \frac{\lfloor y \rfloor}{y^s} - \frac{\lfloor x \rfloor}{x^s} + s\int_x^y \frac{\lfloor u \rfloor}{u^{s+1}} \mathrm{d}u,$$

where $\lfloor \cdot \rfloor$ is the integer part function.

3. Deduce that for $\sigma > 1$ and $x \geq 1$,

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^{+\infty} \frac{\{u\}}{u^{s+1}} \mathrm{d}u,$$

where $\{\cdot\}$ is the fractional part function.

4. Deduce another proof of the analytic continuation of ζ to $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > 0\}$.

   Remark. By integrating by parts multiple times, or using the Euler-Maclaurin summation formula, one can obtain the analytic continuation of ζ to any half-plane of the form $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > -k\}$, with $k \in \mathbb{N}$.
5. Prove that $\frac{1}{\sigma-1} < \zeta(\sigma) < \frac{\sigma}{\sigma-1}$ for $\sigma > 0$. In particular, $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.
6. Let $\delta > 0$. Prove that

$$\zeta(s) = \frac{1}{s-1} + O(1)$$

for $\delta \leq \sigma \leq 2, |t| \leq 3$.

7. Now assume $|t| \geq 3$ and take $x = |t|$ in the result of 3.
   (a) Show that
   $$\left| \sum_{n \leq x} \frac{1}{n^s} \right| \leq 1 + \int_1^x \frac{\mathrm{d}u}{u^\sigma}$$
   for $\sigma \geq 0$.
   (b) Show that for $\sigma \geq 1 - \frac{c}{\log x}$ (where $c \geq 0$ is a fixed constant),
   $$\int_1^x \frac{\mathrm{d}u}{u^\sigma} = O(\log x).$$
   (c) Deduce that
   $$\zeta(s) = O(\log |t|)$$
   for $\sigma \geq \max\left(\delta, 1 - \frac{c}{\log |t|}\right)$, $|t| \geq 3$.

Remark. In the same manner, we prove
$$\zeta'(s) = \frac{-1}{(s-1)^2} + O(1)$$
for $\delta \leq \sigma \leq 2$, $|t| \leq 3$ and
$$\zeta'(s) = O(\log^2 |t|)$$
for $\sigma \geq \max\left(\delta, 1 - \frac{c}{\log |t|}\right)$, $|t| \geq 3$.

Exercise 5. [A first non-vanishing result]

1. Show that for every $\theta \in \mathbb{R}$, $2(1 + \cos\theta)^2 = 3 + 4\cos\theta + \cos(2\theta)$.
2. Let $\sigma > 1$ and $t \in \mathbb{R}$. Show that
   $$3 \log \zeta(\sigma) + 4\mathfrak{Re}(\log(\zeta(\sigma + it))) + \mathfrak{Re}(\log(\zeta(\sigma + 2it))) \geq 0$$
   and deduce that
   $$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$
3. Prove by contradiction that $\zeta(1 + it) \neq 0$ for every $t \neq 0$.

Remark. With some work, one can show that this non-vanishing is actually **equivalent** to the prime number theorem, without an error term. In the next exercise sheet we will show that a wider zero-free region for ζ implies a corresponding error term in the prime number theorem.