Units

Exercise 1. [Fundamental units]

1. What number fields have a group of units of rank 1 ?
2. Let $K = \mathbb{Q}(\sqrt{d})$ with $d > 1$ square-free. Show that there exists a unique generator $u = a + b\sqrt{d}$ of the free part of $\mathcal{O}_K^\times$ with positive a and b. We call this unit the *fundamental unit* of K.
3. Assume $d \equiv 2, 3 \bmod 4$ and let b be the smallest positive integer such that $db^2 + 1$ or $db^2 - 1$ is a square, and call it a^2 with $a > 0$. Show that $a + b\sqrt{d}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$.
4. Compute the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for $d = 2, 3, 6, 7, 10, 11$.
5. Assume $d \equiv 1 \bmod 4$ and let b be the smallest positive integer such that $db^2 + 4$ or $db^2 - 4$ is a square, and call it a^2 with $a > 0$. Show that $\frac{a+b\sqrt{d}}{2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$.
6. Compute the fundamental unit of $\mathbb{Q}(\sqrt{d})$ for $d = 5, 13, 17, 21$.

Exercise 2. [Regulator of a number field]

1. Let $M = (m_{i,j})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{R})$ with $m_{i,i} > 0, m_{i,j} < 0$ and $\sum_{k=1}^n m_{i,k} = 0$ for $1 \le i \ne j \le n$. Show that any family of $n-1$ columns of M is linearly independent over $\mathbb{R}$.
2. Let $M \in \mathcal{M}_{n-1,n}(\mathbb{R})$ with rows summing to zero. Prove that all the minors of size $n-1$ of M are equal.
3. Let K be a number field and $u_1, \dots, u_{r_1+r_2-1}$ be a fundamental system of units of $\mathcal{O}_K$, *i.e.* a basis of the free part of $\mathcal{O}_K^\times$. The regulator of K is
$$R_K = |\det(\dim_{\mathbb{R}}(\sigma_i(K)) \log|\sigma_i(u_j)|).$$

Exercise 3. [Fundamental unit of a cubic field] Let K be a cubic field of signature $(1, 1)$ and let ε be its fundamental unit, *i.e.* a generator of the free part of $\mathcal{O}_K^\times$ such that $\varepsilon > 1$. We will show that
$$\varepsilon^2 > \frac{|\Delta_K| - 24}{4}.$$

1. Prove that $K = \mathbb{Q}(\varepsilon)$ and $N_{K/\mathbb{Q}}(\varepsilon) = 1$.
2. Let ε_2 and $\overline{\varepsilon_2}$ be the conjugates of ε over $\mathbb{Q}$, and $u \in \mathbb{R}$ such that $\varepsilon = u^2$. Show that $\varepsilon_2 = u^{-1}\exp(-i\theta)$ with $0 \le \theta \le \pi$.
3. Show that $\sqrt{|\mathrm{disc}(\varepsilon)|} = 4(a - \cos\theta)\sin\theta$ with $2a = u^3 + u^{-3}$.
4. Let $g = 2X^2 - aX - 1$ and let ρ be a root of g satisfying $|\rho| \le 1$. Show that $\sqrt{|\mathrm{disc}(\varepsilon)|} \le 4(a - \rho)\sqrt{1 - \rho^2}$.
5. Show that there exists a unique root ρ of g such that $-1 \le \rho \le -\frac{1}{2u^3}$.
6. Show that $|\mathrm{disc}(\varepsilon)| < 4u^6 + 24$ and conclude.

7. Let $\alpha = \mathbb{Q}(\sqrt[3]{2})$. Show that the fundamental unit of $\mathbb{Q}(\alpha)$ is $1 + \alpha + \alpha^2$.

Exercise 4. [Cyclotomic units]

Let $n \geq 3$ and $K = \mathbb{Q}(\zeta_n)$ with $\zeta_n = e^{\frac{2i\pi}{n}}$. Let

$$I = \{k \in \mathbb{N} \,|< k < n/2, \gcd(k,n) = 1\}\,.$$

1. Give a condition on $k \in \mathbb{Z}/n\mathbb{Z}$ for $\xi_k = \frac{1-\zeta_n^k}{1-\zeta_n}$ to be a unit in $\mathcal{O}_K$.
2. Show that for all $k \in (\mathbb{Z}/n\mathbb{Z})^\times$,

$$\zeta_n^{\frac{1-k}{2}} \xi_k = \pm \frac{\sin(k\pi/n)}{\sin(\pi/n)}.$$

3. Deduce a relation between ξ_k and ξ_{n-k} up to a root of unity and an upper bound on the rank of the group generated by the ξ_k. Compare it to the rank of $\mathcal{O}_K^\times$.
4. Let K^+ be the maximal real subfield of K. What is its degree over $\mathbb{Q}$? Describe its embeddings and compute the rank of $\mathcal{O}_{K^+}^\times$.
5. Prove that every ξ_k is, up to a n^{th}-root of unity, a unit of $\mathcal{O}_{K^+}$.

Remark. One can show that if n is a prime power then the subgroup generated by the ξ_k has finite index in $\mathcal{O}_K^\times$. Moreover, this index is h_{K^+}.