

Localization (2)

Exercise 1. Let I be an ideal of A and $\pi : A \rightarrow A/I$ the projection. Let S be a multiplicative subset of A and $T = \pi(S)$. Give an isomorphism between $S^{-1}A/S^{-1}I$ and $T^{-1}(A/I)$.

Exercise 2. Let A be a ring, S a multiplicative subset of A , and M be an A -module. Show that the following properties are equivalent:

1. for all $s \in S$, multiplication by s is an automorphism of M .
2. the natural map $M \rightarrow S^{-1}M$ is an isomorphism.
3. the module M can be endowed of an $S^{-1}A$ -module structure compatible with its A -module structure (and then the $S^{-1}A$ -module structure is unique).

Exercise 3. Let A be a commutative ring, M a finitely generated A -module, and S a multiplicative subset of A . Assume that $S^{-1}M = 0$. Show that there exists $s \in S$ such that $sM = 0$. Is this still true if M is not finitely generated?

Exercise 4. Let A be an integral domain and M an A -module. Let $T(M)$ be the submodule of torsion elements.

1. Show that $T(M)$ is the kernel of the map $M \rightarrow (A \setminus \{0\})^{-1}M$.
2. Show that for any multiplicative subset S of A , we have $S^{-1}T(M) = T(S^{-1}M)$.
3. Show that the following conditions are equivalent:
 - (a) M is torsion-free
 - (b) for any prime ideal \mathfrak{p} of A , $M_{\mathfrak{p}}$ is torsion-free
 - (c) for any maximal ideal \mathfrak{m} of A , $M_{\mathfrak{m}}$ is torsion-free

Exercise 5. Let A be a commutative ring, and let M be an A -module. Show that the following are equivalent:

- (i) M is flat;
- (ii) for any prime ideal \mathfrak{p} of A , the A -module $M_{\mathfrak{p}}$ is flat;
- (iii) for any maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$ is flat.

Exercise 6. Let M and N be two A -modules, with N finitely generated. Let $u : M \rightarrow N$. Show that u is surjective if and only if for all maximal ideals \mathfrak{m} of A , $u : M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$ is surjective.

Give an example to show that this is not true with "surjective" replaced by "injective".

Exercise 7. Let A be a commutative ring and S be a multiplicative subset of A .

1. We define $S_{\text{sat}} = \{s \in A \mid \exists a \in A, sa \in S\}$.
 - (a) Check that S_{sat} is a multiplicative subset of A and that $(S_{\text{sat}})_{\text{sat}} = S_{\text{sat}}$.
 - (b) Show that the rings $S_{\text{sat}}^{-1}A$ and $S^{-1}A$ are naturally isomorphic.
2. Now assume that A is an Euclidean domain with an Euclidean function N .
 - (a) Show that for any $s \in S_{\text{sat}}$ and any irreducible element $p \in A$ dividing s , we have $p \in S_{\text{sat}}$.
 - (b) Show that the map defined by $N'(a) = \min_{x \in A \setminus \{0\}} N(ax)$ is an Euclidean function on A that satisfies $N'(a) \leq N'(ab)$ for every $a, b \in A \setminus \{0\}$.

- (c) Show that one can define a function on $S_{\text{sat}}^{-1}A$ by $M\left(\frac{ta}{s}\right) = N'(a)$ where $t, s \in S_{\text{sat}}$ and $a \in A$ with all its irreducible factors in $A \setminus S_{\text{sat}}$.
- (d) Deduce that $S^{-1}A$ is an Euclidean domain.

Exercise 8. Let $(f_i)_{i \in I}$ be a family of elements of A that generate the unit ideal. We say that a family of elements $(a_i)_{i \in I}$ with $a_i \in A[1/f_i]$ is compatible if for all $i, j \in I$, the images of a_i and a_j in $A[1/f_i f_j]$ are the same. Show that if $(a_i)_{i \in I}$ is a compatible family, then there exists a unique $a \in A$ such that a_i is the image of a in $A[1/f_i]$. (Hint: start with the case where I has only two elements).

Exercise 9. Let $\varphi : A \rightarrow B$ be a ring homomorphism. Let \mathfrak{q} be a prime ideal of B and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.

1. Check that there exists a ring homomorphism $\varphi_{\mathfrak{q}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ that sends $\frac{a}{b}$ onto $\frac{\varphi(a)}{\varphi(b)}$.
2. Is the map $\varphi_{\mathfrak{q}}$ surjective (resp. injective) when φ is?