ENS de Lyon — Master 1 – Algèbre avancée
TD8 — 2020-2021

Local rings, localization (1)

Exercise 1.

1. Is every subring of a local ring necessarily a local ring?

2. Let A be a ring and I an ideal of A. Describe the ideals of A/I, the prime ideals of A/I, the maximal ideals of A/I. Is every quotient of a local ring necessarily a local ring?

3. Let$\mathfrak{m}$ be a maximal ideal of A and $n \geq 1$ a positive integer. Show that the ring $A/\mathfrak{m}^n$ is local.

4. Show that the following are equivalent

   (a) The ring A is local;

   (b) the set $A \setminus A^{\times}$ is an ideal of A;

   (c) for any $a, b \in A$ such that $a + b = 1$, we have $a \in A^{\times}$ or $b \in A^{\times}$.

Exercise 2. Let A be a commutative ring and S a multiplicative subset of A. Consider the ideal $I = \{x \in A, sx = 0 \text{ for some } s \in S\}$. Show that if the image of any element of S is invertible in A/I, then $A/I = S^{-1}A$.

1. Let A and B be rings, and $S = \{(1,0),(1,1)\} \subset A \times B$. Compute $S^{-1}(A \times B)$.

2. Describe $(\mathbb{Z}/6\mathbb{Z})_{(2)}$ (localization with respect to a prime ideal).

3. Let $n \geq 2$, and $m \neq 0$ be integers. Describe $(\mathbb{Z}/n\mathbb{Z})[1/m]$.

Exercise 3. Let A be an integral domain and K its field of fractions.

1. Show that for any multiplicative subset S of A that does not contain 0, the ring $S^{-1}A$ is naturally a subring of K.

2. Describe the following localizations with respect to a prime ideal: $k[X]_{(X-a)}$, $\mathbb{Z}_{(p)}$.

3. What are $\bigcap_{\mathfrak{p} \text{ prime}} A_{\mathfrak{p}}$ and $\bigcap_{\mathfrak{m} \text{ maximal}} A_{\mathfrak{m}}$?

Exercise 4. Let A be a commutative ring and $f \in A$. Let $S_f = \{f^n, n \geq 0\}$. Show that $S_f^{-1}A$ is isomorphic to $A[X]/(fX - 1)$.

Exercise 5. Let A be a commutative ring.

1. Let S be a multiplicative subset of A. Show that $S^{-1}A \neq 0$ if and only if $0 \notin S$. Deduce that $A[1/f] \neq 0$ if and only if f is not nilpotent.

2. Let $\mathcal{N}(A)$ be the ideal of nilpotent elements of A. Show that $\mathcal{N}(A)$ is the intersection of all prime ideals of A.

Exercise 6. Let $S \subset T$ be multiplicative subsets of A, and $\phi_S : A \to S^{-1}A$ the natural map. Show that $T^{-1}A$ and $\phi_S(T)^{-1}(S^{-1}A)$ are naturally isomorphic.

Describe $(\mathbb{Z}^2)_{\mathbb{Z} \times \{0\}}$ (localization with respect to a prime ideal).

Exercise 7. Let A be a commutative ring, and S a multiplicative subset of A. Let M and N be two $S^{-1}A$-modules. Show that the natural map $\mathrm{Hom}_{S^{-1}A}(M,N) \to \mathrm{Hom}_A(M,N)$ is bijective. Deduce that $M \otimes_A N$ and $M \otimes_{S^{-1}A} N$ are canonically isomorphic.

Exercise 8. Let A be a commutative ring, and let S_0 be the set of elements of A that are not zero divisors.

1. Show that S_0 is a multiplicative subset of A, and that the canonical map $A \to S_0^{-1}A$ is injective. Show that for any element $x \in S_0^{-1}A$, either x is a unit or it is a zero divisor. The ring $S_0^{-1}A$ is called the total ring of fractions of A.

2. Let S be a multiplicative subset of A. Show that the map $A \to S^{-1}A$ is injective if and only if $S \subset S_0$.

3. Compute $S_0^{-1}A$ in the following cases: A is finite, $A = \mathbb{Z}^2$.

Exercise 9.

1. Let n be a positive integer. When is the ring $\mathbb{Z}/n\mathbb{Z}$ local ?

   From now on we fix p a prime number.

2. For every $m \geq n \geq 1$, denote by
$$\pi_{n,m} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$$
   the canonical reduction. We define
$$\mathbb{Z}_p = \left\{ (a_n)_n \in \prod_n \mathbb{Z}/p^n\mathbb{Z} \mid \pi_{n,m}(a_m) = a_n \text{ for all } m \geq n \geq 1 \right\}.$$
   Show that $(a_n)_n \in \mathbb{Z}_p$ if and only if $\pi_{n,n+1}(a_{n+1}) = a_n$. Prove that $\mathbb{Z}_p$ is a subring of the product $\prod_n \mathbb{Z}/p^n\mathbb{Z}$ and that the natural projection $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ is surjective for all $n \geq 1$. The ring $\mathbb{Z}_p$ is called the *ring of p-adic integers*.

3. Show that the natural ring homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ is injective. Moreover, prove that $\mathbb{Z}_p$ is an integral domain.

4. Show that an element $(a_n)_n \in \mathbb{Z}_p$ is a unit if and only if $a_1 \neq 0$. Deduce that $\ker \pi_1$ is the unique maximal ideal of $\mathbb{Z}_p$ and therefore that it is a local ring.

5. Show that $\mathbb{Z}_p$ is a principal ideal domain whose ideals are generated by powers of p.