ENS de Lyon
TD7

Master 1 – Algèbre avancée
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Modules over a principal ideal domain

Exercise 1. Find the invariant factors of the $\mathbb{Z}$-module

$$M = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

Exercise 2. How many isomorphism classes of abelian groups of order 24 are there?

Exercise 3. Compute the image and the kernel of the matric

$$C = \begin{pmatrix} 1 & 4 & 0 & 3 \\ 0 & 3 & 9 & 12 \\ -1 & -1 & 3 & 3 \end{pmatrix}.$$

Exercise 4. Let A be a principal ideal domain.

1. Let M be a finitely generated A-module. Describe $\mathrm{Ann}_A(M) = \{a \in A \mid \forall x \in M, ax = 0\}$ in terms of the invariant factors of M.
2. Let G be a finite abelian group, and $d > 0$ such that $(d) = \mathrm{Ann}_{\mathbb{Z}}(G)$. Show that $\mathrm{Card}(G) \geq d$, with equality if and only if G is cyclic.
3. Deduce from this that any finite subgroup of the group of units of a field is cyclic.

Exercise 5. Compute the similarity invariants of the following matrices with coefficients in $\mathbb{C}$:

$$\begin{pmatrix} -8 & 1 & 5 \\ 2 & -3 & -1 \\ -4 & 1 & 1 \end{pmatrix}; \quad \begin{pmatrix} 0 & 2 & -1 \\ 2 & -4 & 3 \\ 2 & -6 & 4 \end{pmatrix}; \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Exercise 6. What are the similarity invariants of : an homothety ? a transvection ? a diagonalizable endomorphism with distinct eigenvalues ? a projection ? a Jordan block ?

Exercise 7. Let A be a ring, and I and ideal of A. Show that the only endomorphisms of the A-module A/I are multiplication by a for some a in A.

Let E be a finite-dimensional K-vector space and u an endomorphism of E. Show that u is cyclic if and only if any endomorphism of E that commutes with u is a polynomial in u.

Exercise 8. We say that an A-module M is indecomposable if it is nonzero and if for any submodules P and Q of M, the equality $M = P \oplus Q$ implies $P = 0$ or $Q = 0$.

1. Let A be a principal ideal domain. Prove that the indecomposable finitely generated A-modules are, up to isomorphism, A and the A-modules $A/(p^n)$, where $n \in \mathbb{N} \setminus \{0\}$ and p is an irreducible element of A.
2. Give an example of indecomposable, non-finitely generated A-module.

Exercise 9. We say that an A-module M is simple if it is nonzero and has no non-trivial submodule. We say that an A-module M is semi-simple if and only if it is nonzero and for any submodule P of M, there exists a submodule Q such that $P \oplus Q = M$.

1. Prove that any simple module is finitely generated, and indecomposable. Is the reciprocal true ?
2. Prove that if M is noetherian and semi-simple, then it is a finite sum of simple modules.

3. Let A be a principal ideal domain. Prove that any simple A-module is of the form $A/(p)$ for some irreducible element p of A. Deduce that any semi-simple, finitely generated A-module is of the form $\oplus_{i=1}^n A/(p_i)$ for some irreducible elements p_i of A. Conversely, show that any such module is semi-simple.

4. Let K be a field, and let u be an endomorphism of K^n. We say that u is semi-simple if and only if any stable subspace has a stable complement. Show that u is semi-simple if and only if u endows K^n with a structure of semi-simple $K[X]$-module. Deduce that u is semi-simple if and only if its minimal polynomial is a product of distinct irreducible polynomials. Does u necessarily become diagonalizable over the algebraic closure of K?

Exercise 10. Let A be a principal ideal domain, and M a finitely generated torsion A-module. For p an irreducible element of A, let $M(p)$ be the p-primary part of M.

Show that for each irreducible element p of A, there exists an element $a_p \in A$ such that $a_p M = M(p)$ and multiplication by a_p induces the identity of $M(p)$ and $a_p M(q) = 0$ for any irreducible element q that is not equivalent to p.

Let K be an algebraically closed field, and u an endomorphism of a finite-dimensional K-vector space E. For $\lambda \in K$, let $E_\lambda = \ker(u - \lambda \operatorname{Id})^{\dim E}$. Show that the projector with image E_λ and kernel $\oplus_{\mu \neq \lambda} E_\mu$ is a polynomial in u.

Exercise 11. Let K be an algebraically closed field, and E a finite-dimensional K-vector space. Show that u and v are conjugated if and only if for all $\lambda \in K$ and all $n > 0$, $(u - \lambda \operatorname{Id})^n$ and $(v - \lambda \operatorname{Id})^n$ have the same rank.

Exercise 12. (Finite $\mathbb{Z}[i]$-modules)

1. Recall why the ring $\mathbb{Z}[i]$ is a PID.

2. Up to isomorphism, how many $\mathbb{Z}[i]$-modules have 3 elements ? 5 elements ? 9 elements ?

3. Let p be a prime number such that $p \equiv 1 \mod 4$. Provide a $\mathbb{Z}[i]$-module structure on $\mathbb{Z}/p\mathbb{Z}$.

4. Let $a + ib \in \mathbb{Z}[i]$. What is the cardinality of $\mathbb{Z}[i]/(a + ib)$?

5. Deduce that an odd prime number p in $\mathbb{Z}$ is a sum of two squares if, and only if, $p \equiv 1 \mod 4$.

6. What are the prime elements of $\mathbb{Z}[i]$? For each prime p of $\mathbb{Z}[i]$, describe the ring $\mathbb{Z}[i]/(p)$.

Exercise 13. (A Bezout ring that is not principal) Let U be a connected open subset of $\mathbb{C}$. Denote by $\mathcal{H}(U)$ the ring of holomorphic functions on U, and by $\mathcal{M}(U)$ the ring of meromorphic functions on U.

We recall the following theorems from complex analysis:

Theorem 1 (Weierstrass theorem) *Let A be a subset of U with no accumulation point in U, and for all $a \in A$ let $m_a \in \mathbb{Z}_{>0}$. Then there exists $f \in \mathcal{H}(U)$ that has a zero of order exactly m_a at a for each $a \in A$, and no zero outside A.*

Theorem 2 (Mittag-Leffler theorem) *Let A be a subset of U with no accumulation point in U, and for all $a \in A$ let $m_a \in \mathbb{Z}_{>0}$, and elements $c_{1,a}, \ldots, c_{m_a,a}$ in $\mathbb{C}$. Then there exists $f \in \mathcal{M}(U)$ with principal part $\sum_{i=1}^{m_a} c_{i,a}(z-a)^{-i}$ at each $a \in A$ and no pole outside A.*

1. Let f and g be in $\mathcal{H}(U)$ with no common zero in U. Show that there exist u, v in $\mathcal{H}(U)$ such that $uf + vg = 1$ (Hint: find $F, G \in \mathcal{M}(U)$ such that fF, gG and $F + G - 1/fg$ are holomorphic).

2. Let f and g be in $\mathcal{H}(U)$. Show that there exists $h \in \mathcal{H}(U)$ such that $(f, g) = (h)$. Deduce that any finitely generated ideal of $\mathcal{H}(U)$ is principal (we say that $\mathcal{H}(U)$ is a Bezout ring).

3. Show that $\mathcal{H}(U)$ is not noetherian, hence not a principal ideal domain*.

*In fact, one could prove that $\mathcal{H}(U)$ is an elementary divisor ring but this is quite long.