Tensor products of algebras, Galois theory, modules over principal ideal domains

In the following, A is a commutative ring.

Exercise 1. Describe the following $\mathbb{Q}$-algebras:

$$\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{R}, \quad \mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{R}, \quad \mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{R}, \quad \mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i).$$

Exercise 2. Let K be a field and L/K a finite extension of degree n. Denote by $\overline{K}$ an algebraic closure of K.

1. Show that the following are equivalent:
   (i) the extension L/K is separable;
   (ii) for any extension M/K, the algebra $L \otimes_K M$ does not contain nonzero nilpotent elements;
   (iii) the algebra $L \otimes_K \overline{K}$ does not contain nonzero nilpotent elements.
2. Show that L/K is a Galois extension if, and only if, the algebras $L \otimes_K L$ and L^n are isomorphic.

Exercise 3. Let A be a ring and B a commutative A-algebra. Let M and N be B-modules. We see them as A-modules via the map $A \to B$.

1. Show that we have a natural injective map: $\Phi_{M,N} : \mathrm{Hom}_B(M, N) \to \mathrm{Hom}_A(M, N)$. Show that this map is bijective for all B-modules M and N if $B = A/I$ for some ideal I of A, or if A is a domain and $B = \mathrm{Frac}(A)$.
2. Show that there exists a canonical surjective A-linear map $f_{M,N} : M \otimes_A N \to M \otimes_B N$ sending $m \otimes n$ to $m \otimes n$.

   We can define two B-module structures on $M \otimes_A N$, via the action of B on M or on N.
3. Assume that for all B-modules M and N, the map $\Phi_{M,N}$ is bijective. Show that for all B-modules M and N, $f_{M,N}$ is an isomorphism, the two B-module structures on $M \otimes_A N$ are the same, and $f_{M,N}$ is B-linear.
4. Give an example where $f_{M,N}$ is not an isomorphism, and an example where the two B-module structures on $M \otimes_A N$ are not the same.

Exercise 4. Let $A \to B$ be an homomorphism of commutative rings. Show that for any A-modules M and N, there exists a unique isomorphism of B-modules $B \otimes_A (M \otimes_A N) \simeq (B \otimes_A M) \otimes_B (B \otimes_A N)$ which sends $b \otimes (m \otimes n)$ onto $b((1 \otimes m) \otimes (1 \otimes n))$.

Exercise 5. Let A be an integral domain, and K be its fraction field.

1. Let V and W be two K-vector spaces. Explain why for any $v \in V \setminus \{0\}$ and any $w \in W \setminus \{0\}$, we have $v \otimes w \neq 0$.
2. Let V and W be two K-vector spaces. Show that $V \otimes_A W$ and $V \otimes_K W$ are canonically isomorphic.
3. Let M be an A-module.
   (a) Show that we have an isomorphism of A-modules $K \otimes_A M \simeq K \otimes_A (M/M_{\mathrm{tors}})$.
   (b) Deduce that M_{tors} is the kernel of the A-linear map $M \to K \otimes_A M$ sending $m \in M$ onto $1 \otimes m$.
4. Let M and N be two A-modules. Show that for any elements $m \in M$ and $n \in N$ that are not A-torsion elements, we have $m \otimes n \neq 0$.

Exercise 6. Let M be a noetherian A-module.

1. Show that if A is noetherian, then $M[X]$ is a noetherian $A[X]$-module.
2. Let $\mathrm{Ann}_A(M) = \{a \in A\ ,\ a \cdot m = 0\ \forall m \in M\}$ be the annihilator of M. Show that the ring $B = A/\mathrm{Ann}_A(M)$ is noetherian.
3. Deduce that $M[X]$ is a noetherian $A[X]$-module (we do not assume that A is a noetherian ring).

Exercise 7. Let k be an algebraically closed field and let A, B be two finitely generated k-algebras. In this exercise we show that if A becomes isomorphic to B after base change by some k-algebra C, then they are already isomorphic over k.

1. Let $f : A \otimes_k C \to B \otimes_k C$ be an isomorphism of C-algebras. Show that there exists a k-subalgebra $C' \subset C$, finitely generated over k, such that f is defined over C', i.e. that there exists an isomorphism $f' : A \otimes_k C' \to B \otimes_k C'$ of C'-algebras such that f is the extension of scalars of f'.
2. By reducing modulo a maximal ideal of C', show that f' induces an isomorphism from A to B.

   Hint: you can use the following fact (Nullstellensatz): if k is algebraically closed, any field that is also a finitely generated k-algebra is equal to k.
3. Let M and N be two finitely generated A-modules. Show that if $M \otimes_k C \cong N \otimes_k C$ as $A \otimes_k C$-modules then M is isomorphic to N as A-modules.

   Hint: use the fact that M and N are of finite presentation and the same idea as in the previous point.
4. Give an example of two non isomorphic finitely generated k-algebras which are isomorphic after the extension of scalars $k \to \overline{k}$ (here we do not assume k algebraically closed).