ENS de Lyon — Master 1 – Algèbre avancée
TD5 — 2020-2021

Symmetric and exterior algebras; base change

In the following, A is a commutative ring.

Exercise 1. Let B be an A-algebra, $n \geq 1$ be an integer and M, N be two A-modules.

1. Show that any A-linear map $\varphi : M \to N$ induces canonically an A-linear map $\mathrm{Sym}^n(\varphi) : \mathrm{Sym}^n(M) \to \mathrm{Sym}^n(N)$ and an A-linear map $\Lambda^n(\varphi) : \Lambda^n(M) \to \Lambda^n(N)$.
2. Show that any A-linear map $\varphi : M \to B$ satisfying $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$ induces a unique homomorphism of A-algebras $\mathrm{Sym}(\varphi) : \mathrm{Sym}(M) \to B$.

   Deduce that, if B is commutative, then there is a natural bijection between $\mathrm{Hom}_A(M, B)$ (A-module homomorphisms) and $\mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Sym}(M), B)$ (A-algebra homomorphisms).

Exercise 2. Let A be a ring, and M an A-module. Let $m \geq 1$ be an integer.

1. Show that for all $\sigma \in \mathfrak{S}_m$, there is a linear map $u_\sigma : T^m M \to T^m M$ such that $u_\sigma(x_1 \otimes \cdots \otimes x_m) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}$.

   Let
   $$S^m M = \{x \in T^m M \mid \forall \sigma \in \mathfrak{S}_m, u_\sigma(x) = x\}$$
   and
   $$A^m M = \{x \in T^m M \mid \forall \sigma \in \mathfrak{S}_m, u_\sigma(x) = \varepsilon(\sigma)x\}.$$
2. Show that $S^m M$ and $A^m M$ are submodules of $T^m M$.
3. Assume that $m!$ is invertible in A. Show that the natural projections $S^m M \to \mathrm{Sym}^m M$ and $A^m M \to \Lambda^m M$ are isomorphisms. Hint: introduce the m-linear maps
   $$s : M^m \to T^m M,\ (x_1, \ldots, x_m) \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}$$
   and
   $$a : M^m \to T^m M,\ (x_1, \ldots, x_m) \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}.$$

Exercise 3. Let M, N be two A-modules.

1. Show that for any $n \geqslant 0$, there is a natural isomorphism of A-modules: $\mathrm{Sym}^n(M \oplus N) \simeq \bigoplus_{k=0}^{n} \Big(\mathrm{Sym}^k(M) \otimes_A \mathrm{Sym}^{n-k}(N)\Big)$, and show that the A-algebras $\mathrm{Sym}(M \oplus N)$ and $\mathrm{Sym}(M) \otimes \mathrm{Sym}(N)$ are isomorphic.
2. Show that for any integer $n \geqslant 0$, there is a natural isomorphism of A-modules $\Lambda^n(M \oplus N) \simeq \bigoplus_{k=0}^{n} \big(\Lambda^k M \otimes_A \Lambda^{n-k} N\big)$.

Exercise 4. Let M be a free A-module of rank n, and $f \in \mathrm{End}_A(M)$. Let $a(f) = \det(X \,\mathrm{Id} - f) \in A[X]$, and $b(f) = \sum_{i=0}^{n} (-1)^i X^{n-i} \operatorname{tr}(\Lambda^i f) \in A[X]$. Our goal is to show that $a(f) = b(f)$.

1. Show that the identity holds if A is an algebraically closed field.
2. Let $u : B \to C$ be a ring homomorphism, and f a B-linear map $B^n \to B^n$. Assume that the equality holds for f. Show that it also holds for $\mathrm{Id}_C \otimes f \in \mathrm{End}_C(C \otimes_B M)$. If u is injective, show that the equality holds for $\mathrm{Id}_C \otimes f$ if and only if it holds for f.

3. Deduce that the equality holds for any ring A and $f \in \mathrm{End}_A(A^n)$. Hint: introduce the ring $B = \mathbb{Z}[(T_{i,j})_{1\leq i,j\leq n}]$ and the map $B^n \to B^n$ with matrix $T = (T_{i,j})_{1\leq i,j\leq n}$.

Exercise 5.

1. Let M be an A-module and I an ideal of A. Show that M/IM is naturally endowed with a A/I-module structure, which coincides with $A/I \otimes_A M$.
2. Assume A is integral and let K be its fraction field. Let M be an A-module such that for any $a \in A \setminus \{0\}$, the multiplication-by-a map is an automorphism of M. Show that M is naturally endowed with a K-vector space structure, which coincides with $K \otimes_A M$.
3. Let M be an A-module and $M[X]$ be the additive group of polynomials with coefficients in M. Provide an $A[X]$-module structure on $M[X]$ so that $M[X]$ and $M \otimes_A A[X]$ are isomorphic as $A[X]$-modules.

Exercise 6. Let M be an A-module and $A \to B$ be a homomorphism of commutative rings.

1. Show that if M is projective, then the B-module $B \otimes_A M$ is projective.
2. Show that if M is finitely generated, then the B-module $B \otimes_A M$ is finitely generated.
3. Show that if M is finitely presented, then the B-module $B \otimes_A M$ is finitely presented.
4. Find an example of A-module M such that the scalar restriction to A of the B-module $B \otimes_A M$ is not isomorphic to M.

Exercise 7. Let $A \to B$ be an homomorphism of commutative rings. Show that for any A-modules M and N, there exists a unique isomorphism of B-modules $B \otimes_A (M \otimes_A N) \simeq (B \otimes_A M) \otimes_B (B \otimes_A N)$ which sends $b \otimes (m \otimes n)$ onto $b((1 \otimes m) \otimes (1 \otimes n))$.