Tensor product

Exercise 1. Let G be a finitely generated abelian group, seen as a $\mathbb{Z}$-module.

1. Assume that G is finite. Let H be a finite abelian group such that G and H have coprime orders. Show that $G \otimes_{\mathbb{Z}} H = 0$.
2. Let m, n be positive integers. Compute $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$.
3. Show that if G is of exponent m, then $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} G$ is a finite abelian group of exponent $\gcd(n, m)$.
4. Show that $G \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ if, and only if, G is finite. Deduce an example of $\mathbb{Z}$-modules M and N having submodules M' and N' such that the map
$$M' \otimes_{\mathbb{Z}} N' \to M \otimes_{\mathbb{Z}} N$$
is not injective.
5. Show that $\mathrm{Id}_G \otimes 1 : G \to G \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective if and only if G is free.

Exercise 2. Let n be a positive integer. Describe the following tensor products of $\mathbb{Z}$-modules:
$$\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}, \quad \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}.$$

Exercise 3. Show that a free module is flat. Deduce that a projective module is flat.

Exercise 4. Let M_1, N_1, M_2, N_2 be four A-modules. Consider the homomorphism of A-modules:
$$h : \mathrm{Hom}_A(M_1, N_1) \otimes_A \mathrm{Hom}_A(M_2, N_2) \to \mathrm{Hom}_A(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$$
defined in the lecture. Provide examples of a commutative ring A and of A-modules M_1, M_2, N_1, N_2 for which the map h is not surjective (resp. is not injective).

Exercise 5. Let N_1, N_2 be two A-submodules of an A-module N, and let M be a flat A-module.

1. Show that there exists a short exact sequence of A-modules:
$$0 \longrightarrow N_1 \cap N_2 \longrightarrow N \longrightarrow (N/N_1) \oplus (N/N_2)$$
2. Show that, as A-submodules of $N \otimes_A M$, the modules $(N_1 \cap N_2) \otimes_A M$ and $(N_1 \otimes_A M) \cap (N_2 \otimes_A M)$ are equal.

Exercise 6. Let M and N be two A-modules. Let $\sum_i x_i \otimes y_i \in M \otimes N$ be such that $\sum_i x_i \otimes y_i = 0$. Show that there exists finitely generated submodules M' of M and N' of N such that $x_i \in M'$ for all i, $y_i \in N'$ for all i, and $\sum_i x_i \otimes y_i = 0$ as an element of $M' \otimes N'$.

Deduce that: if there exists a family (M_i) of submodules of M such that each M_i is flat over A, and such that any finitely generated submodule of M is contained in one of the M_i, then M is flat over A.

Let A be an integral domain and K its fraction field. Show that K is flat over A.

Exercise 7. Let k be a positive integer and M be a nonzero A-module. We denote the A-module $\underbrace{M \otimes_A \cdots \otimes_A M}_{k \text{ terms}}$ by $M^{\otimes k}$ and $M^{\otimes 0} = A$.

1. Show that $M^{\otimes k+1}$ is isomorphic to $M^{\otimes k} \otimes_A M$.
2. Assume that M is finitely generated and let $(e_1, \ldots, e_d)$ be a generating family such that the submodule N of M generated by $(e_1, \ldots, e_{d-1})$ is not equal to M.

(a) Show that $I = \{a \in A,\ a \cdot e_d \in N\}$ is a proper ideal of A and that A/I is isomorphic to M/N.

(b) Define a nonzero A-multilinear map $M^k \to A/I$ that sends $(e_d, \dots, e_d)$ onto $1 \mod I$.

(c) Deduce that $M^{\otimes k}$ is nonzero.

3. Give an example of a non finitely generated module M for which $M^{\otimes k} = 0$ for any $k \geq 2$.

4. Let n be a positive integer and take $A = \mathbb{Z}$. Compute $(\mathbb{Z}/n\mathbb{Z})^{\otimes k}$.

5. Provide an example of a module M and a submodule N of M such that for all $k \geq 2$, the A-module $N^{\otimes k}$ is not isomorphic to any submodule of $M^{\otimes k}$.

Exercise 8. Let X be a compact Hausdorff topological space and Y be a normed $\mathbb{R}$-vector space. Show that the canonical $\mathbb{R}$-linear map $\mathrm{C}^0(X, \mathbb{R}) \otimes_{\mathbb{R}} Y \to \mathrm{C}^0(X, Y)$ is injective, and that its image is the subspace of continuous functions $f : X \to Y$ such that $\mathrm{Im}(f)$ is contained in a finite-dimensional subspace of Y. Deduce that $\mathrm{C}^0(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{C}^0(X, \mathbb{C})$.

Exercise 9. Let $A = \mathbb{Z}[X]$ and $I = (2, X)$.

1. Show that $2 \otimes X - X \otimes 2 \neq 0$ in $I \otimes_A I$.
   Hint: One can note that evaluation on even integers of polynomials in I is an even integer.

2. Show that $2 \otimes X - X \otimes 2$ is of 2-torsion and of X-torsion.

3. Show that the A-submodule of $I \otimes_A I$ generated by $2 \otimes X - X \otimes 2$ is isomorphic to A/I.

Exercise 10.* Let A be a commutative ring and M be an A-module. We want to show that M is flat if (and only if) for all finitely generated ideal I of A, the map

$$I \otimes_A M \to M$$

is injective. Assume that the latter is true.

1. Show that for all ideal I of A, the map $I \otimes_A M \to M$ is injective.

2. We show by induction on n that if K is a submodule of A^n, then the map $K \otimes_A M \to M^n$ is injective; n=1 is the previous question; assume the result to be true for n, show that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \cap A & \longrightarrow & K & \longrightarrow & K/K \cap A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A^{n+1} & \longrightarrow & A^n & \longrightarrow & 0 \end{array}$$

   with exact rows and conclude. (Here $K \cap A$ is the intersection of K with the submodule generated by $(1, 0, \dots, 0)$).

3. Let N be a finitely generated A-module and P an A-module. Assume that $N \to P$ is injective. Show that $N \otimes M \to P \otimes N$ is injective (Hint: a different snake).

4. Show that M is flat.