ENS de Lyon — Master 1 – Algèbre avancée
TD1 — 2020-2021

Modules

All rings are assumed to be commutative.

Exercise 1. (Some examples of modules)

1. Let A be a ring. Describe submodules of A seen as a A-module.
2. Describe $\mathbb{Z}$-modules.
3. Let K be a field. Describe $K[X]$-modules, their submodules and linear maps.

Exercise 2. (Universal properties) Let A be a ring.

1. Let M be a A-module and N a submodule of M. Show that the quotient M/N satisfies the following universal property : for any A-module M' and any A-linear map $f : M \longrightarrow M'$ such that $N \subset \ker f$, there exists a unique A-linear map $\tilde{f} : M/N \longrightarrow M'$ such that f factorizes through $\tilde{f}$, *i.e.* $f = \tilde{f} \circ \pi_N$, where $\pi_N : M \twoheadrightarrow M/N$ is the canonical projection (make a diagram representing the situation). Deduce that if P is a submodule of N, then $(M/P)/(N/P)$ is isomorphic to M/N.
2. Let $\{M_i \mid i \in I\}$ be a family of A-modules. Show that the direct sum $\oplus_{i\in I} M_i$ satisfies the following universal property : for any A-module N and any family $\{f_i : M_i \longrightarrow N \mid i \in I\}$ of A-linear maps, there exists a unique A-linear map $g : \oplus_{i\in I} M_i \longrightarrow N$ such that $g_{|M_i} = f_i$ for every $i \in I$. Draw a diagram in the case where I is finite and $M_i = A$?
3. Let $\{M_i \mid i \in I\}$ be a family of A-modules. Show that the product $\prod_{i\in I} M_i$ satisfies the following universal property : for any A-module N and any family $\{f_i : N \longrightarrow M_i \mid i \in I\}$ of A-linear maps, there exists a unique A-linear map $g : N \longrightarrow \prod_{i\in I} M_i$ such that $g_i = f_i$ for every $i \in I$. Draw a diagram in the case where I is finite and $M_i = A$?
4. Deduce from the last two points that if $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ are families of A-modules, then $\mathrm{Hom}_A(\oplus_{i\in I} M_i, \prod_{j\in J} N_j) \simeq \prod_{(i,j)\in I\times J} \mathrm{Hom}_A(M_i, N_j)$.

Exercise 3. Find two non-isomorphic $\mathbb{Z}$-modules M_1, M_2 such that there exists an exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to M_i \to \mathbb{Z}/2\mathbb{Z} \to 0$ for $i = 1, 2$.

Exercise 4. (Not as easy as linear algebra) Let A be a ring and M be a free A-module.

1. If $(x_i)_{i\in I}$ is a generating fmily of M, does it contain a basis of M ?
2. If $(x_i)_{i\in I}$ is a linearly independent family of M, can it be extended to a basis of M ? Does every submodule of M admit a direct sum complement ?
3. Show that, if $n > 1$, $\mathbb{Z}/n\mathbb{Z}$, seen as a $\mathbb{Z}$-module, does not contain any linearly independent family. Conclude that $\mathbb{Z}/n\mathbb{Z}$ is not a free $\mathbb{Z}$-module.

Exercise 5. (Dual module) If M is an A-module, we set $M^\vee = \mathrm{Hom}_A(M, A)$.

1. Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.
2. If M is a torsion module over the ring A, that is if $M_{\mathrm{tors}} = M$, and A is an integral domain, show that $M^\vee = 0$.
3. Show that there is a natural map $M \to M^{\vee\vee}$. Show that this map is an isomorphism when A is a field and M a finite-dimensional vector space. Give an example where this map is not injective, and an example where it is not surjective.

Exercise 6. (Torsion) A module M is called a torsion module if $M_{\mathrm{tors}} = M$, and torsion-free if $M_{\mathrm{tors}} = \{0\}$.

1. What are the torsion elements of the A-module A ?

   We assume for the rest of the exercise that A is an integral domain.

2. Show that M_{tors} is a submodule of M and that M/M_{tors} is a torsion-free A-module. Does it still hold if A is not an integral domain ?

3. Let N be a submodule of M. Express N_{tors} in terms of M_{tors}. Deduce that M_{tors} is a torsion module.

4. Let N and P be two submodules of M such that $M = N \oplus P$. Show that $M_{\text{tors}} = N_{\text{tors}} \oplus P_{\text{tors}}$. Deduce that A^r is torsion-free.

5. Let (M_i) be a family of A-modules. Show that $(\oplus_i M_i)_{\text{tors}} = \oplus_i (M_{i,\text{tors}})$ but that the inclusion $(\prod_i M_i)_{\text{tors}} \subset \prod_i (M_{i,\text{tors}})$ is not necessarily an equality.

6. Prove that if the sequence $0 \to M \to N \to P$ is exact, then so is $0 \to M_{\text{tors}} \to N_{\text{tors}} \to P_{\text{tors}}$.

7. Show that there exists a unique A-linear map $\tilde{f}$ such that the following diagram is commutative (here π_M and π_N denote canonical surjections),

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M/M_{\text{tors}} & \xrightarrow{\ \tilde{f}\ } & N/N_{\text{tors}} \end{array}$$

Exercise 7. (Annihilators) Let A be a ring and M be a A-module.

1. Set $\text{Ann}(M) = \{a \in A \mid \forall m \in M, a \cdot m = 0\}$. Show that $\text{Ann}(M)$ is an ideal of A and that M admits a natural structure of $A/\text{Ann}(M)$-module.

2. Let $x \in M$, and set $\text{Ann}(x) = \{a \in A \mid a \cdot x = 0\}$. Show that $\text{Ann}(x)$ is an ideal of A, and that the submodule $A \cdot x$ of M is isomorphic to $A/\text{Ann}(x)$. Deduce that $A \cdot x$ is free if and only if x is not a torsion element of M.

Exercise 8. Let A be a ring and I an ideal of A. Show that I is a free submodule of A if and only if I is principal, generated by a non-zero divisor of A. Give an example of a submodule of a free module which is not free.

Exercise 9. Let A be a ring such that every A-module is free. Show that A is a field.

Exercise 10.* (The Baer-Specker group $\mathbb{Z}^{\mathbb{N}}$)

1. For any $n \in \mathbb{N}$, set $e_n = (0, \dots, 0, \underbrace{1}_{n}, 0, \dots) \in \mathbb{Z}^{\mathbb{N}}$. Let us show that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \simeq \mathbb{Z}^{(\mathbb{N})}$:

   (a) Give a natural $\mathbb{Z}$-linear map $\mathbb{Z}^{(\mathbb{N})} \hookrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$.

   (b) Let $f \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$. Let us show that $f(e_n) = 0$ for every sufficiently large n. If not, show that there exists a sequence $(d_n)_{n \in \mathbb{N}}$ such that no integer x satisfies $x \equiv \sum_{i=0}^{N-1} 2^i d_i f(e_i) \bmod 2^N$ for every $N \in \mathbb{N}$, and consider $S = \sum_{n \in \mathbb{N}} 2^n d_n e_n \in \mathbb{Z}^{\mathbb{N}}$. Hint : Use a diagonal argument.

   (c) Let $x \in \mathbb{Z}^{\mathbb{N}}$. Show that for any $n \in \mathbb{N}$, there exist $a_n, b_n \in \mathbb{Z}$ such that $x_n = 2^n a_n + 3^n b_n$. Deduce that if $f \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ vanishes on $\mathbb{Z}^{(\mathbb{N})}$ then $f(x) = 0$. Conclude.

2. Show that $\mathbb{Z}^N$ is not a free $\mathbb{Z}$-module. Hint : use Exercise 2.