ÉNS de Lyon Master 1 – Algèbre avancée
Mini-DM 4 2020-2021

Local and almost local properties

Let A be a commutative ring and M an A-module. We say a property $\mathcal{P}$ of M (or of an A-linear map u) is local when $\mathcal{P}$ holds for M (or u) if and only if for every prime ideal $\mathfrak{p}$ of A, $\mathcal{P}$ holds for the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ (or the $A_{\mathfrak{p}}$-linear map $u_{\mathfrak{p}}$). Recall (or admit the fact) that being the zero-module, injectivity/surjectivity or flatness are local properties.

In algebraic geometry, localizing at a prime ideal corresponds to working locally around a given point, hence the term local.

1. Assume $A = \mathbb{F}_2^{\mathbb{N}}$ and let $\mathfrak{a}$ be its ideal $\mathbb{F}_2^{(\mathbb{N})}$.
   (a) Let $\mathfrak{p}$ be a prime ideal of A. Show that any $x \in A_{\mathfrak{p}}$ satisfies $x^2 = x$.
   (b) Recall that $A_{\mathfrak{p}}$ is a local ring. Show that $A_{\mathfrak{p}} \simeq \mathbb{F}_2$.
   (c) Show that for every prime ideal $\mathfrak{p}$ of A, the $A_{\mathfrak{p}}$-module $\mathfrak{a}_{\mathfrak{p}}$ is finitely generated, yet $\mathfrak{a}$ is not finitely generated. Thus, being finitely generated is not a local property.
   (d) Show that for every prime ideal $\mathfrak{p}$ of A, the $A_{\mathfrak{p}}$-module $A_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$ is free, yet $A/\mathfrak{a}$ is not projective (*Hint* : use the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$). Thus, being free and being projective are not local properties.

   Remark : The same example shows that being noetherian is not a local property.

2. We say a property $\mathcal{P}$ of M (or of an A-linear map u) is almost local when $\mathcal{P}$ holds for M (or u) if and only if there exist $a_1, \ldots, a_n, s_1, \ldots, s_n \in A$ such that $a_1 s_1 + \cdots + a_n s_n = 1$ such that $\mathcal{P}$ holds for each A_{s_i}-module M_{s_i} (or each of the A_{s_i}-linear map u_{s_i}).

   Show that a local property which is preserved by taking localization is almost local. In particular, being a surjective linear map is an almost local property (you can decide to only prove this last statement).

3. We will show that being a noetherian module is an almost local property. The forward direction is immediate by taking $n = 1, a = s = 1$, so we prove the converse.
   (a) Assume first that each M_{s_i} is finitely generated. Find an integer r and a linear map $u : A^r \longrightarrow M$ such that each $u_{s_i} : A_{s_i}^r \longrightarrow M_{s_i}$ is surjective, and conclude that M is finitely generated.
   (b) Show that if each M_{s_i} is noetherian, then so is M.

   Remark : When every non-zero element of A is contained in finitely many maximal ideals of A, being a noetherian A-module actually is a local property by a theorem of Nagata.