ENS de Lyon
Mini-DM 2

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Equational criterion of flatness

Let A be a (commutative) ring. We let $\otimes$ be the tensor product of A-modules.

Let M be an A-module. A relation $\sum_{i=1}^{n} a_i x_i = 0$ in M (with $a_i \in A$ and $x_i \in M$) is *trivial* if it comes from relations in A i.e. if there is an integer m and a matrix $(b_{ij}) \in M_{n,m}(A)$ such that for all j, $\sum_{i=1}^{n} a_i b_{ij} = 0$ and there are elements y_j of M such that for all i, $x_i = \sum_{j=1}^{m} b_{ij} y_j$

1. The goal of this question is to prove the equational criterion of flatness. This will give a more concrete caracterisation of flatness. The criterion is the following:

   Let M be an A-module, then M is flat if and only if all relations are trivial in M.

   (a) Assume that M is flat. Take a relation $\sum_{i=1}^{n} a_i x_i = 0$ with $a_i \in A$ and $x_i \in M$. Let I be the ideal generated by $a_1, \dots, a_n$. Show that the element $\sum_{i=1}^{n} a_i \otimes x_i$ of $I \otimes M$ is zero.

   (b) Let e_i be the canonical basis of A^n. Let K be the kernel of the morphism $A^n \to I$ sending e_i to a_i. Show that there is an element of $K \otimes M$ mapping to $\sum_{i=1}^{n} e_i \otimes x_i$. Conclude that if M is flat, all relations are trivial in M.

   (c) Assume that all relations are trivial in M. Let I be a finitely generated ideal of A and let $\sum_{i=1}^{n} a_i \otimes x_i$ be an element of $I \otimes M$ which is sent to 0 in $A \otimes M = M$. Show that $\sum_{i=1}^{n} a_i \otimes x_i = 0$. Conclude.

2. Let k be a field and assume $A = k[x, y]$. Let M be the ideal of A generated by x and y. Is M flat over A ?

3. Assume that A is a local ring. Let $\mathfrak{m}$ be its maximal ideal and $k = A/\mathfrak{m}$. Let M be a finitely generated flat A-module. We want to show that M is free. Let $\overline{M} = M/\mathfrak{m}M$ and let $(\overline{u_1}, \dots, \overline{u_n})$ be a free family of $\overline{M}$ as a k-vector space.

   (a) We will proceed by induction on n to show that $(u_1, \dots, u_n)$ is free. Show that if $n = 1$, (u_1) is free.

   (b) Assume the result for $n-1$. Let $\sum_{i=1}^{n} a_i u_i = 0$ be a relation in M. Show that a_n is a linear combination of $a_1, \dots, a_{n-1}$. Deduce that $(u_1, \dots, u_n)$ is free.

   (c) Show that if $(\overline{u_1}, \dots, \overline{u_n})$ is a generating family of $\overline{M}$, $(u_1, \dots, u_n)$ is a generating family of M. Conclude.